DECOMPOSITION OF BIRATIONAL TORIC MAPS IN BLOW-UPS AND BLOW-DOWNS

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ABSTRACT. We prove that a toric birational map between two complete smooth toric varieties of the same dimension can be decomposed in a sequence of equivariant blow-ups and blow-downs along smooth centers.

INTRODUCTION

Two questions, concerning factorization of birational morphisms and maps by blow-ups along smooth varieties, are of fundamental importance in birational algebraic geometry.

Let X' and X'' be complete algebraic varieties which are birationally equivalent.

(1) Does there exist a third variety X and birational morphisms $X \longrightarrow X'$, $X \longrightarrow X''$, which are compositions of blow-ups along closed irreducible subvarieties?

(2) Does there exist a sequence of varieties X_i, for $i = 0, \ldots, n$, such that $X_0 = X'$, $X_n = X''$, and X_{i+1} is obtained from X_i by a blow-up or blow-down along a closed irreducible subvariety?

So far an affirmative answer to (1) was given for the surface case by Italian mathematicians (see for instance [3]). In higher dimensions nothing in general is known, except the famous **Hironaka's theorem on elimination of points of indeterminacy**, which states that for any pair X', X'' of birationally equivalent complete complex algebraic varieties there is a finite sequence of blow-ups in smooth closed irreducible subvarieties $X \longrightarrow X'$, such that there exists a birational morphism $g : X \longrightarrow X''$ [5].

The present work gives a positive answer to (2) in the case of smooth toric varieties. The toric version of question (2) appears in [7] and [8], and is called the weak Oda conjecture. So far the conjecture has been solved by Danilov in the case when the dimension is ≤ 3 [2]. There is a theorem due to M.Reid in [9], closely related to the above problem, which shows a way of breaking down a birational toric morphism of varieties with mild singularities into elementary steps. The elementary steps, however, are more complicated than blow-ups and blow-downs.

Toric varieties have proven their value as a testing ground for investigating birational morphisms and maps. (See for instance [6], [9].) The behaviour of toric varieties permits us to understand, at least partially, the general picture of arbitrary algebraic varieties. On the other hand, the combinatorial theorems obtained here

can be directly applied to a larger class of objects, such as toroidal embeddings [6] or spherical spaces.

The present paper is divided into 3 parts. In the preliminaries we introduce the main notions of fan theory. Then, using the fan language, we formulate the main combinatorial theorem of the paper (Theorem A). In Part 1 we talk about general properies of complexes. We introduce basic definitions and notation used through the whole paper. In Part 2 we prove the main theorem for simplicial complexes (Theorem 8.1). Then we associate to any simplicial fan a simplicial complex and deduce from Theorem 8.1 the simplicial case of Theorem A. In Part 3 we deal with complexes equipped with some integral structure, corresponding to the lattice structure on the fan. We also complete the proof of Theorem A.

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After this paper was submitted for publication I learned that a similar result was obtained by Robert Morelli.

0. PRELIMINARIES

0. Fans: Basic Properties and Notations.
Formulation of the Main Theorem in the Fan Language

We recall briefly the notation from the theory of fans. Fix the rational vector space $\mathbf{Q}^n$, containing the lattice $\mathbf{Z}^n$.

For any vectors (v_i), $i = 1, \ldots, k$, in $\mathbf{Q}^n$ by $\langle v_1, \ldots, v_k \rangle$ we shall denote the cone spanned by these vectors.

By a *fan* we shall understand the finite set Σ of convex cones in $\mathbf{Q}^n$ provided:

1. Every cone $\delta \in \Sigma$ can be written in the form $\delta = \langle e_1, \ldots e_k \rangle$, where $e_i \in \mathbf{Z}^n \subset \mathbf{Q}^n$.
2. Every face τ of a cone $\delta \in \Sigma$ belongs to Σ.
3. Cones are adjacent along their faces.
4. $\{0\} \in \Sigma$.

A fan Σ is *regular* iff every cone $\delta \in \Sigma$ can be represented in the form $\delta = \langle e_1, \ldots, e_k \rangle$, where $\{e_i\}$ is a subset of a basis of $\mathbf{Z}^n$. A fan is *simplicial* iff every cone belonging to it is simplicial. A *support* of a fan Σ is the set $|\Sigma| = \bigcup_{\delta \in \Sigma} \delta$. A *decomposition* of a fan Σ is another fan Σ' such that $|\Sigma| = |\Sigma'|$ and $\forall_{\delta' \in \Sigma'} \exists_{\delta \in \Sigma} \delta' \subset \delta$. A decomposition Σ' of a fan Σ is *regular* iff it is a decomposition and a regular fan. A fan Σ in $\mathbf{Q}^n$ is *complete* if $|\Sigma| = \mathbf{Q}^n$.

There is a bijection between the fans in $\mathbf{Q}^n \supset \mathbf{Z}^n$ and the normal torus embeddings of the n-dimensional torus T [8, Theorem 1.4]. In the above correspondence the decompositions are associated with the T-equivariant proper morphisms (identical on T) [8, Corollary 1.17]. The regular fans are associated with the smooth torus embeddings [8, Theorem 1.10], and the simplicial fans are in correspondence with the Q-factorial embeddings ([9]). The complete fans correspond to the complete torus embeddings [8, Theorem 1.11] (see also [2], [6], [7]).

Let Σ be a fan in $\mathbf{Q}^n$. Let v be any vector in $\mathbf{Q}^n$. By the *blow-up at the ray* $\langle v \rangle$ we mean the decomposition which changes every cone δ of Σ containing $\langle v \rangle$ to the set of the convex hulls $\sigma + v$, where σ is a face of δ which does not contain $\langle v \rangle$. The inverse transformation will be called a *blow-down*. We shall say that a blow-up is *regular* iff both fans, i.e. before the blow-up and after the blow-up, are regular. The inverse transformation of a regular blow-up is called a *regular blow-down.*

Assume that Σ is a regular fan. One can show that a blow-up at the primitive vector v is regular iff vector v can be written in the form $v = b_1 + \ldots + b_k$ for some cone $\delta \in \Sigma$ such that $\delta = \langle b_1, \ldots, b_k \rangle$, where b_i, $i = 1, \ldots, k$, is a subset of a basis of $\mathbf{Z}^n$ [8, Proposition 1.26].

In the above mentioned correspondence blow-ups are associated with some blow-ups along T-equivariant subshemes (which can be non-reduced). The regular blow-ups are in $1 - 1$ correspondence with the blow-ups along closed T-equivariant submanifolds.

The main theorem of this paper is the following:

Theorem A. *Let Σ', Σ'' be two fans in $\mathbf{Q}^n$, such that $|\Sigma'| = |\Sigma''|$. Assume that Σ', Σ''" are regular (resp. simplicial). Then there is a sequence of regular (resp. simplicial) fans Σ_i for $i = 0, \ldots, n$, such that $\Sigma_0 = \Sigma'$, $\Sigma_n = \Sigma''$ and each Σ_i is obtained from Σ_{i-1} by a blow-up or blow-down for $i = 1, \ldots, n$ (which is regular if Σ' and Σ'' are regular).*

We prove this theorem for simplicial fans in Part 2, and for regular fans in Part 3. Via the correspondence between fans and torus embeddings we can deduce from this theorem the following result.

Theorem B. *Let X' and X'' be smooth embeddings of torus T. Assume that either*

   i) *X', X'' are complete*

or

   ii) *there is an equivariant proper morphism $\Phi : X' \longrightarrow X''$.*

Then there exists a sequence of smooth torus embeddings X_i, for $i = 0, \ldots, n$, such that $X_0 = X'$, $X_n = X''$, and X_{i+1} is obtained from X_i by a T-equivariant blow-up or blow-down along a closed smooth T-equivariant irreducible subvariety.

PART I

1. COMPLEXES. BASIC PROPERTIES AND NOTATIONS

We will consider in this work geometric objects of the affine space $\mathbf{Q}^n \subset \mathbf{R}^n$, where $\mathbf{Q}$ is the set of the rational numbers in $\mathbf{R}$.

We adopt all of the definitions and notations from the theory of convex bodies and complexes in affine real space to the case of the field $\mathbf{Q}$. Thus we are going to use the standard notions like *simplex, polytope, vertex, face, convex hull* (conv), *affine hull* (aff), *relative interior* (int), *relative boundary* (∂), of the polytopes. (For the definitions see [4].)

By $\Delta(A_0, \ldots, A_n)$ we shall denote the simplex with vertices $A_0, \ldots, A_n$. By $P = \mathrm{conv}(A_0, \ldots, A_n)$ we mean the polytope which is a convex hull of points $A_0, \ldots, A_n$. By $\mathrm{aff}(X)$ we denote the affine hull of a set X. In particular $\mathrm{aff}(A_0, \ldots, A_n)$ denote the affine hull of the points $A_0, \ldots, A_n$.

By a *complex* in $\mathbf{Q}^d$, we understand here a finite family $\mathbf{C}$ of polytopes in $\mathbf{Q}^d$ such that:

   i) *every face of member of $\mathbf{C}$ is itself a member of $\mathbf{C}$*, and
   ii) *the intersection of any two members of $\mathbf{C}$ is a face of each of them.*

We shall call a member of a complex $\mathbf{C}$ a *face* of $\mathbf{C}$. A complex $\mathbf{C}$ will be called *simplicial* provided all its members are simplices. By a *subcomplex* $\mathbf{C}'$ of $\mathbf{C}$ we will understand a subset of $\mathbf{C}$ satisfying property i) from the above definition. By the

support of a complex $\mathbf{C}$ we mean the set $|\mathbf{C}| = \bigcup_{\mathbf{P} \in \mathbf{C}} \mathbf{P}$. By a *subdivision* of a complex $\mathbf{C}$ we shall mean another complex $\mathbf{C}'$ provided:

i) *for every face there exists a face $\rho' \in \mathbf{C}'$ of $\mathbf{C}$ such that $\rho' \subset \rho$ and*

ii) $|\mathbf{C}'| = |\mathbf{C}|$.

We shall say that a subdivision $\mathbf{C}'$ of $\mathbf{C}$ is *proper* if the set of vertices of $\mathbf{C}$ equals the set of vertices of $\mathbf{C}'$: $\mathrm{vert}(\mathbf{C}') = \mathrm{vert}(C)$. By a *triangulation* of $\mathbf{C}$ we mean a subdivision which is a simplicial complex. By a *proper triangulation* we shall mean a triangulation which is simultaneously a proper subdivision.

Consider a complex $\mathbf{C}$, its subdivision $\mathbf{S}$, and a subcomplex $\mathbf{L}$ of $\mathbf{C}$. It is clear that $\mathbf{S}$ determines a subdivision of $\mathbf{L}$, which will be called a *restriction* of $\mathbf{S}$ to $\mathbf{L}$ and denoted by $\mathbf{S}_{|\mathbf{L}}$.

We say that a face of a complex is *maximal* iff it is maximal with respect to the relation of face. For a complex $\mathbf{C}$ denote by $\mathbf{C}_{\max}$ the set of maximal faces of $\mathbf{C}$. Note that $\mathbf{C}_{\max}$ completely determines $\mathbf{C}$.

Let $\mathbf{C}$ be a complex, and $\mathbf{P}$ a face in $\mathbf{C}$. The *star* of $\mathbf{P}$ in $\mathbf{C}$ is the set $\mathrm{Star}(\mathbf{P}; \mathbf{C})$, of all faces of $\mathbf{C}$ containing $\mathbf{P}$. We denote by $\mathrm{CompStar}(\mathbf{P}; \mathbf{C})$ the complex determined by $\mathrm{Star}(\mathbf{P}; \mathbf{C})$. The *antistar* $\mathrm{Ast}(\mathbf{P}; \mathbf{C})$ of $\mathbf{P}$ in $\mathbf{C}$ is the complex consisting of all faces of $\mathbf{C}$ which are disjoint from $\mathrm{int}\,\mathbf{P}$. Note that $\mathrm{Star}(\mathbf{P}; \mathbf{C}) \cup \mathrm{Ast}(\mathbf{P}; \mathbf{C}) = \mathbf{C}$.

By a *complex pair* or simply a *pair* $(\mathbf{C}; \Omega)$, we understand a pair consisting of a complex $\mathbf{C}$, and a finite subset Ω of $|\mathbf{C}|$. We will call Ω the set of *added vertices*. We define a set of vertices of the pair $(\mathbf{C}; \Omega)$ as the set $\mathrm{vert}(\mathbf{C}; \Omega) := \mathrm{vert}\,\mathbf{C} \cup \Omega$. By a *triangulation* (or resp., a *subdivision*) of the *pair* $(\mathbf{C}; \Omega)$ we mean a triangulation $\mathbf{T}$ (resp. subdivision $\mathbf{T}$) of the complex $\mathbf{C}$ such that $vert\mathbf{T} = vert(\mathbf{C}; \Omega)$. For example a triangulation of the pair $(\mathbf{C}; \emptyset)$ is just a proper triangulation.

By a *polytope* (resp. a *simplex*) of the pair $(\mathbf{C}; \Omega)$ we mean any polytope $\mathbf{P}$ (resp., simplex $\mathbf{P}$) such that $\mathbf{P}$ is contained in some face of $\mathbf{C}$ and $\mathrm{vert}\,\mathbf{P} = \mathrm{vert}(\mathbf{C}; \Omega) \cap \mathbf{P}$.

2. Blow-Ups and Elementary Transformations

By the *blow-up* of a complex $\mathbf{K}$ at a point $P \in |\mathbf{K}|$ we mean a subdivision of $\mathbf{K}$ which is obtained from $\mathbf{K}$ by replacing any polytope $\mathbf{P} \in \mathbf{K}$ containing P by the set of convex hulls of P and all faces of $\mathbf{P}$ which do not contain P. By the blow-up of a complex $\mathbf{K}$ at a point $P \notin |\mathbf{K}|$ we mean the complex $\mathbf{K}$.

The inverse transformation to a blow-up is called a *blow-down*.

We denote the blow-up of a complex $\mathbf{K}$ at P by $\mathbf{K}_{(P)}$. Note that if $\mathbf{K}$ is simplicial then $\mathbf{K}_{(P)}$ is also. By $\mathbf{K}_{(P_1 \dots P_n)}$ we mean successive blow-ups of $\mathbf{K}$ at points $P_1 \dots, P_n$ i.e.

$$\mathbf{K}_{(P_1 \dots P_n)} = \big((\mathbf{K}_{(P_1)}) \dots \big)_{(P_n)}$$

We call $\mathbf{K}_{(P_1 \dots P_n)}$ a *blow-up of $\mathbf{K}$ at the sequence of points $P_1 \dots, P_n$* .

Definition 2.1. By a *basic polytope* $\mathbf{P}^B = \mathrm{conv}(A_0, ..., A_n; B_0, \dots, B_m)$, where $n, m \geq 1$, we mean an $n + m$-dimensional polytope $\mathrm{conv}(A_0, \dots, A_n, B_0, \dots, B_m)$, in $\mathbf{Q}^k$, such that

$$(1) \qquad \sum_{i=0}^{n} a_i A_j = \sum_{j=0}^{m} b_j B_j \quad \text{for some } a_i, b_j > 0; \quad \sum_{i=0}^{n} a_i = \sum_{j=0}^{m} b_j = 1.$$

By an *elementary polytope*

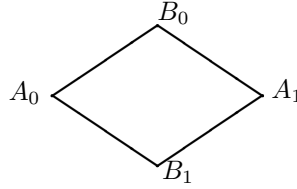$$\mathbf{P}^E = \mathrm{conv}(A_0, \dots, A_n; B_0, \dots, B_m : C_1, \dots, C_r)$$

we mean an $(n + m + r)$-dimensional polytope

$$\operatorname{conv}(A_0, \dots, A_n,\ B_0, \dots, B_m,\ C_1, \dots, C_r), \quad \text{where} \quad n \geq 1, \quad m \geq 1,$$

such that:

$$(1') \qquad \sum_{i=0}^{n} a_i A_i = \sum_{j=0}^{m} b_j B_j \quad \text{for some } a_i, b_i > 0; \quad \sum_{i=0}^{n} a_i = \sum_{j=0}^{m} b_j = 1.$$

Example 1.

Note that (1) (or (1′)) is the only relation in the set of vertices of the basic (or elementary) polytope. Let us call each proper triangulation of a basic or an elementary polytope a *canonical triangulation*.

Lemma 2.2. *Each basic polytope* $\mathbf{P}^B = \operatorname{conv}(A_0, \dots, A_n; B_0, \dots, B_m)$ *(or elementary polytope* $\mathbf{P}^E = \operatorname{conv}(A_0, \dots, A_n; B_0, \dots, B_m : C_1, \dots, C_r)$*) has exactly two canonical triangulations. These are* $\mathbf{A}$ *and* $\mathbf{B}$*, such that*

$$\mathbf{A}_{\max} = \{\Delta(A_0, \dots, A_n, B_0, \dots, \check{B}_i, \dots, B_m) : \quad i = 0, \dots, m\},$$
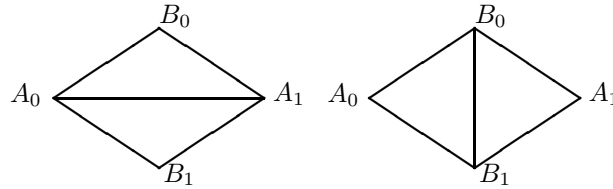
$$\mathbf{B}_{\max} = \{\Delta(A_0, \dots, \check{A}_j, \dots, A_n,\ B_0, \dots, B_m) : \quad j = 0, \dots, n\},$$

or respectively

$$\mathbf{A}_{\max} = \{\Delta(A_0, \dots, A_n, B_0, \dots, \check{B}_i, \dots, B_m, C_1, \dots, C_r)\}_{i=0,\dots,m},$$

$$\mathbf{B}_{\max} = \{\Delta(A_0, \dots, \check{A}_j, \dots, A_n\ B_0, \dots, B_m, C_1, \dots, C_r)\}_{j=0\dots,n}.$$

Example 2.

Proof. It suffices to consider the case of a basic polytope. First note that $\mathbf{A} = \mathbf{P}^B{}_{(A_i)}$ for any A_i, and $\mathbf{B} = \mathbf{P}^B{}_{(B_i)}$ for any B_j. Thus $\mathbf{A}$ and $\mathbf{B}$ are in fact triangulations.

Now note that every $n + m$-dimensional simplex of a proper triangulation is a simplex of $\mathbf{A}_{\max}$ or $\mathbf{B}_{\max}$. On the other hand each simplex from $\mathbf{A}_{\max}$ has the face $\Delta(A_0, \dots, A_n)$, and each simplex from $\mathbf{B}_{\max}$ has the face $\Delta(B_0, \dots, B_m)$. The face $\Delta(A_0, \dots, A_n)$ intersects $\Delta(B_0, \dots, B_m)$ at P. Thus there cannot be a triangulation which contain simplices of $\mathbf{A}_{\max}$ and $\mathbf{B}_{\max}$. $\square$

The basic polytope $\mathbf{P}^B = \mathrm{conv}(A_0, \dots, A_n; B_0, \dots, B_m)$, as is easily seen, is a face of the elementary polytope $\mathbf{P}^E = \mathrm{conv}(A_0, \dots, A_n; B_0, \dots, B_m; C_1, \dots, C_r)$. Triangulations $\mathbf{A}$, $\mathbf{B}$ of $\mathbf{P}^B$ determine triangulations $\mathbf{A}$, $\mathbf{B}$ of $\mathbf{P}^E$.

Definition 2.3. Let $\mathbf{K}$ be a complex.

(a) An *elementary subdivision* of $\mathbf{K}$ is any subdivision $\mathbf{S}^E$, such that:
   - there is a basic polytope $\mathbf{P}^B$ which is a face of $\mathbf{S}^E$,
   - every face belonging to $\mathrm{Star}(\mathbf{P}^B; \mathbf{S}^E)$ is an elementary polytope, and
   - every face belonging to $\mathrm{Ast}(\mathbf{P}^B; \mathbf{S}^E)$ is a simplex.
(b) The complex $\mathbf{C}^B = \mathrm{CompStar}(\mathbf{P}^B; \mathbf{S}^E)$ is called the *basic complex* of the elementary subdivision.
(c) Any proper triangulation of an elementary subdivision $\mathbf{S}^E$ is called a *canonical triangulation*.
   By Lemma 2.2. there are exactly two canonical triangulations of $\mathbf{S}^E$, which differ only on $\mathrm{CompStar}(\mathbf{P}^B; \mathbf{S}^E)$, where they are induced by the canonical triangulations of $\mathbf{P}^B$.
(d) An elementary transformation of an elementary subdivision changes one of its canonical triangulations to the other. We then say that the two triangulations *differ by an elementary transformation*.
(e) Two triangulations are *elementarily equivalent* or *el-equivalent* iff one of them can be obtained from the other by a sequence of elementary transformations.

From the definitions we deduce that an elementary subdivision determines the two inverse elementary transformations and, conversely, any elementary transformation uniquely determines an elementary subdivision. Thus we will talk about the basic polytopes and canonical triangulations of elementary transformations instead of the basic polytopes and canonical triangulations of elementary subdivisions.

Definition 2.4. Let $\mathbf{K}$ be a complex, and let $\mathbf{T}_0$, $\mathbf{T}_1$ denote two triangulations. We call these triangulations *equivalent* if one of them can be obtained from the other by a sequence of blow-ups and blow-downs.

Proposition 2.5. *Let $\mathbf{K}$ be a complex and let $\mathbf{T}_0$, $\mathbf{T}_1$ be two triangulations. Assume that $\mathbf{T}_0$ and $\mathbf{T}_1$ are el-equivalent. Then they are equivalent.*

Proof. It suffices to show that any elementary transformation can be obtained by a blow-up and a blow-down.

Let $\mathbf{C}^B$ be the basic complex of our elementary transformation. By definition:

$$\mathbf{C}^B_{\max} = \{\mathrm{conv}(A_0, \dots, A_n; B_0, \dots, B_m : C_1^i, \dots, C_{m(i)}^i); i \in I\}.$$

Triangulations $\mathbf{T}_0$ and $\mathbf{T}_1$ coincide outside $\mathbf{C}^B$. Their restrictions to $\mathbf{C}^B$ are two canonical triangulations $\mathbf{A}$, $\mathbf{B}$ such that:

$$
\begin{aligned}
\mathbf{A}_{\max} &= \Big\{\Delta(A_0, \dots, A_n, B_0, \dots, \check{B}_j, \dots, B_m, C_1^i, \dots, C_{m(i)}^i), \\
&\qquad\qquad j = 0, \dots, m, \ i \in I\Big\}, \\
\mathbf{B}_{\max} &= \Big\{\Delta(A_0, \dots, \check{A}_j, \dots, A_n, B_0, \dots, B_m, C_1^i, \dots, C_{m(i)}^i), \\
&\qquad\qquad j = 0, \dots, n, \ i \in I\Big\}.
\end{aligned}
$$

Assume that $\mathbf{T}_0$ on $\mathbf{C}^B$ is $\mathbf{A}$, and $\mathbf{T}_1$ on $\mathbf{C}^B$ is $\mathbf{B}$. Let $P := \Delta(A_0, \ldots, A_n) \cap \Delta(B_0, \ldots, B_m)$. Then $\mathbf{T}_{0(P)}$ and $\mathbf{T}_{1(P)}$ coincide outside $\mathbf{C}^B$. On $\mathbf{C}^B$ both equal

$$\{\Delta(A_0, \ldots, \check{A}_r, \ldots, A_n, B_0, \ldots, \check{B}_j, \ldots, B_m, C_1^i, \ldots, C_{m(i)}^i, P)$$

$$j = 0, \ldots, m, \ \ r = 0, \ldots, n, \ \ i \in I\}.$$

Consequently we can obtain $\mathbf{T}_0$ from $\mathbf{T}_1$ by blowing up at P, and then blowing down at P. $\qquad\square$

3. Inducing Triangulations of a Complex by Triangulations of a Subcomplex.

Let $\mathbf{P} = \mathrm{conv}(A_0, \ldots, A_k, B_0, \ldots, B_m)$ be a polytope such that each B_i is affinely independent of $\{A_0, \ldots, A_k, B_0, \ldots, B_m\} \setminus \{B_i\}$. In particular, the polytope $\mathbf{P}' = \mathrm{conv}(A_0, \ldots, A_k)$ is a face of $\mathbf{P}$. Let $\mathbf{S}$ be a subdivision of $\mathbf{P}'$. Then $\mathbf{S}$ determines a subdivision $\mathbf{P}(\mathbf{S})$ of $\mathbf{P}$ such that $\mathbf{P}(\mathbf{S})_{\max} = \{\mathrm{conv}(\Delta, B_0, \ldots, B_m) : \Delta \in \mathbf{S}_{\max}\}$.

We say that a complex $\mathbf{K}$ *is simplicial with respect to its subcomplex* $\mathbf{R}$ (or *simplicial w.r.t.* $\mathbf{R}$) iff :

a) for every face $\mathbf{k} \in \mathbf{K}$, $\mathbf{k}' = \mathbf{k} \cap |\mathbf{R}|$ is a common face of $\mathbf{k}$ and $\mathbf{R}$, and
b) for every face $\mathbf{k} \in \mathbf{K}$, each vertex P in $\mathrm{vert}(\mathbf{k}) \setminus \mathrm{vert}(\mathbf{k}')$ is affinely independent of the set $\mathrm{vert}(\mathbf{k}) \setminus \{\mathbf{P}\}$.

Examples. A simplicial complex is simplicial w.r.t. any of its faces. Elementary subdivision is simplicial w.r.t. its basic polytope, elementary polytope.

Assume that $\mathbf{C}$ is simplicial w.r.t. its subcomplex $\mathbf{L}$. Then any subdivision $\mathbf{T}$ of $\mathbf{L}$ determines the subdivision $\mathbf{C}(\mathbf{T})$ of $\mathbf{C}$ such that for a polytope $\mathbf{P} \in \mathbf{C}$

$$\mathbf{C}(\mathbf{T})_{|\mathbf{P}} = \mathbf{P}\big(\mathbf{T}_{|(|\mathbf{L}| \cap \mathbf{P})}\big).$$

The statements below follow easily from the definitions.

Proposition 3.1. *If* $\mathbf{T}$ *is a triangulation of* $\mathbf{L}$ *then* $\mathbf{C}(\mathbf{T})$ *is also a triangulation of* $\mathbf{C}$ *(with the previous assumptions).* $\qquad\square$

Proposition 3.2. *Assume that* $\mathbf{C}$ *is simplicial w.r.t.* $\mathbf{L}$. *Let* $\mathbf{P}$ *be a subdivision of* $\mathbf{L}$, *and* $\mathbf{R}$ *be a subdivision of* $\mathbf{P}$, *proper or not. Then* $\mathbf{C}(\mathbf{R})$ *is a subdivision of* $\mathbf{C}(\mathbf{P})$, *respectively proper or not.* $\qquad\square$

Proposition 3.3. *Assume that* $\mathbf{C}$ *is simplicial w.r.t.* $\mathbf{L}$ *and let* $\mathbf{R}$ *be any subdivision of* $\mathbf{L}$. *Then* $\mathbf{C}(\mathbf{R})$ *is simplicial w.r.t.* $\mathbf{R}$. $\qquad\square$

Proposition 3.4. *Assume that* $\mathbf{C}$ *is simplicial w.r.t.* $\mathbf{L}$, *and let* $\mathbf{R}$ *be an elementary subdivision of* $\mathbf{L}$. *Then* $\mathbf{C}(\mathbf{R})$ *is an elementary subdivision of* $\mathbf{C}$. $\qquad\square$

Corollary 3.5. *Let* $\mathbf{C}$ *be a complex simplicial w.r.t. its subcomplex* $\mathbf{L}$. *Let* $\mathbf{T}_0$, $\mathbf{T}_1$ *be two triangulations of* $\mathbf{L}$.

a) *If* $\mathbf{T}_0$, $\mathbf{T}_1$ *differ by an elementary transformation, then* $\mathbf{C}(\mathbf{T_0})$, $\mathbf{C}(\mathbf{T_1})$ *do also.*
b) *If* $\mathbf{T}_0$, $\mathbf{T}_1$ *are el-equivalent, then so are* $\mathbf{C}(\mathbf{T_0})$, $\mathbf{C}(\mathbf{T_1})$.

Proof. Apply Proposition 3.2 to Proposition 3.4. $\qquad\square$

Proposition 3.6. *Assume that* $\mathbf{C}$ *is simplicial w.r.t.* $\mathbf{L}$, *and* $\mathbf{S}$ *is a subdivision of* $\mathbf{L}$. *Let* $W \in |\mathbf{L}|$. *Then* $\mathbf{C}(\mathbf{S})_{(W)} = \mathbf{C}(\mathbf{S}_{(\mathbf{W})})$. $\qquad\square$

Proposition 3.7. *Let* **C** *be simplicial w.r.t.* **L**. *Let* $\mathbf{T}_0$, $\mathbf{T}_1$ *be two equivalent triangulations of* **L**. *Then* $\mathbf{C}(\mathbf{T}_0)$ *and* $\mathbf{C}(\mathbf{T}_1)$ *are also equivalent.*   □

Proposition 3.8. *Let* **C** *be simplicial w.r.t.* **L**. *Let* $\mathbf{R}_0$, $\mathbf{R}_1$ *be two subdivisions of* **L**. *Then* **P** *is a common polytope of* $\mathbf{C}(\mathbf{R}_0)$ *and* $\mathbf{C}(\mathbf{R}_1)$ *iff* $\mathbf{P} \cap \mathbf{L}$ *is a common polytope of* $\mathbf{R}_0$ *and* $\mathbf{R}_1$.   ∎

PART 2

4. El-Equivalence of Some Triangulations

This section is devoted to the proof of the following proposition:

Proposition 4.   A. *Let* Δ *be a simplex. Let* $P, Q \in \Delta$. *Assume no proper face of* Δ *contains both* P *and* Q. *Then* $\Delta_{(P,Q)}$ *and* $\Delta_{(Q,P)}$ *are el-equivalent, and the elementary transformations realizing the above el-equivalence do not change the triangulations on the relative boundary of* Δ.
   B. *Let* $\mathbf{A}, \mathbf{B}$ *be the two canonical triangulations of the basic polytope*

$$\mathbf{P^B} = \mathrm{conv}(A_0, ..., A_n \; ; \; B_0, \dots, B_m).$$

   Let $Q \in \mathbf{P^B}$. *Then* $\mathbf{A}_{(Q)}$ *is el-equivalent to* $\mathbf{B}_{(Q)}$, *and the transformations realizing the el-equivalence do not change the triangulations on the boundary.*
   C. *Let* $\mathbf{A}, \mathbf{B}$ *be the two canonical triangulations of an elementary polytope* $\mathbf{P^E}$. *Let* $Q \in \mathbf{P^E} \backslash \mathbf{P^B}$. *Then* $\mathbf{A}_{(Q)}$ *is el-equivalent to* $\mathbf{B}_{(Q)}$, *and the transformations realizing the el-equivalence do not change the triangulations on the boundary.*

Lemma 4.0. *Let* **C** *be a complex in* $\mathbf{Q}^n$ *and* **T** *be a triangulation. Let* $\mathbf{P^B}$ *be a basic polytope and let* $\{\mathbf{P^E}_i : i \in I\}$ *be a set of elementary polytopes with the same basic polytope* $\mathbf{P^B}$, *such that:*

   1. *For each* $\mathbf{P^E}_i$ *one of its canonical triangulations is a subcomplex of* **T**.
   2. *The triangulation* **B** *of the basic polytope*

$$\mathbf{P^B} = \mathrm{conv}\,(A_0, \dots, A_n \,; B_0, \dots, B_m)$$

   is a subcomplex of **T**. *Then* $\beta = \Delta(B_0, \dots, B_m)$ *is a simplex of* **T** *and for each* α *in* $\mathrm{Star}(\beta; \mathbf{T})$ *there is* $\mathbf{P^E}_i$ *such that* $\alpha \subset \mathbf{P^E}_i$

   Then the set **S** *of all faces of polytopes in* $\mathrm{Ast}(\beta; \mathbf{T}) \cup \{\mathbf{P^E}_i; i \in I\}$ *is an elementary subdivision of* **C**, *and* **T** *is one of the canonical triangulations of this subdivision.*

Proof of Lemma 4.0. $\mathbf{P^E}_i \cap \mathbf{P^E}_j = |\mathbf{T}_{|\mathbf{P^E}_i} \cap \mathbf{T}_{|\mathbf{P^E}_j}|$. Hence $\mathbf{P^E}_i \cap \mathbf{P^E}_j$ is a union of simplices $\delta \in \mathbf{T}_{|P_i^E} \cap \mathbf{T}_{|P_j^E}$. From the definition of an elementary polytope it follows that if $\delta \in \mathbf{T}_{|P_i^E}$ where **T** is a canonical triangulation, then $\mathrm{conv}(\delta; \mathbf{P^B})$ is a face of $\mathbf{P^E}_i$. Thus $\mathbf{P^E}_i \cap \mathbf{P^E}_j$ is a union of common faces $\mathrm{conv}(\delta; \mathbf{P^B})$, and since it is a convex set it must be a single common face of $\mathbf{P^E}_i$ and $\mathbf{P^E}_j$.

Analogously if $\sigma \in \mathrm{Ast}(\beta; \mathbf{T})$ then $\mathbf{P}_i^E \cap \sigma$ is a union of common faces δ of $\mathbf{T}_{|P_i^E}$ and σ. Each such $\delta \in \mathbf{T}_{|P_i^E}$ that does not contain β is a face of $\mathbf{P}_i^E$. Thus $\mathbf{P}_i^E \cap \sigma$ is a union of common faces of $\mathbf{P}_i^E$ and σ, and hence by convexity is a single common face.

Finally, any two faces of $\mathrm{Ast}(\beta; \mathbf{T}) \cup \{\mathbf{P}_i^E : i \in I\}$ are adjacent along their faces. Moreover by property 2 we have that $|\mathrm{Ast}(\beta; \mathbf{T}) \cup \{\mathbf{P}_i^E : i \in I\}| = |\mathbf{C}|$.   ∎

Let $(P; \Omega)$ be a pair, where $\mathbf{P}$ is a polytope and Ω is the set of added vertices. Let $V := \mathrm{vert}(P; \Omega)$. We can represent any subdivision of this pair by the set of its maximal polytopes. On the other hand each polytope of such a subdivision can be represented as the set of its vertices, which is a subset of V. Let $\mathrm{SET}(V)$ denote the set of all subsets of V and $\mathrm{SET}^2(V)$ denote the set of all the subsets of $\mathrm{SET}(V)$. Thus each subdivision of the considered pair determines a unique element of $\mathrm{SET}^2(V)$.

It follows that we can take set-theoretical unions and differences in $\mathrm{SET}^2(V)$. We introduce multiplication in $\mathrm{SET}^2(V) : \beta_0\beta_1 = \{\alpha_i \cup \gamma_j : \alpha_i \in \beta_0, \gamma_j \in \beta_1\}$.

Every $\alpha \in \mathrm{SET}(V)$ determines the element $\{\alpha\}$ from $\mathrm{SET}^2(V)$. Every nonempty element $\alpha \in \mathrm{SET}(V)$ determines the element $\langle\alpha\rangle \in \mathrm{SET}^2(V)$ given by $\langle\alpha\rangle :=$ the set of all $(\mathrm{card}(\alpha) - 1)$ subsets of α.

In particular we have the following properties.

1. If $\alpha, \beta \in \mathrm{SET}(V)$ and $\alpha \cap \beta = \emptyset$, then $\{\alpha \cup \beta\} = \{\alpha\}\{\beta\}$.
2. If $\alpha, \beta \in \mathrm{SET}(V)$ and $\alpha \cap \beta = \emptyset$, then $\langle\alpha \cup \beta\rangle = \langle\alpha\rangle\{\beta\} \cup \{\alpha\}\langle\beta\rangle$.
3. If $\alpha, \beta, \gamma \in \mathrm{SET}^2(V)$, then $\alpha(\beta \cup \gamma) = \alpha\beta \cup \alpha\gamma$.

We shall use these operations and notations in representing triangulations as elements of $\mathrm{SET}^2(V)$.

Further, for $\alpha \in \mathrm{SET}(V)$ denote by $\Delta(\alpha)$ and $\mathrm{conv}(\alpha)$ the simplex and the polytope, respectively, spanned by elements belonging to α. For $\alpha \in \mathrm{SET}^2(V)$, by $\mathrm{conv}(\alpha)$ we mean $\{\mathrm{conv}(\alpha') : \alpha' \in \alpha\}$

Example 1. Let $Q \in \Delta = \Delta(A_0, \dots, A_n)$. Then the triangulation $\Delta_{(Q)}$ is described by $\langle\{A_0, \dots, A_n\}\rangle\{\{Q\}\}$ in $\mathrm{SET}^2(V)$.

Example 2. Let $\mathbf{P^B} = \mathrm{conv}(A_0, \dots, A_n; B_0, \dots, B_m)$ be a basic polytope. Then

$$\mathbf{A} = \{\{A_0, \dots, A_n\}\}\langle\{B_0, \dots, B_m\}\rangle \in \mathrm{SET}^2(V),$$

$$\mathbf{B} = \langle\{A_0, \dots, A_n\}\rangle\{\{B_0, \dots, B_m\}\} \in \mathrm{SET}^2(V).$$

Proof of Proposition 4A. Consider a pair $(\Delta; \{P, Q\})$, where $P, Q \in \Delta$. Let $\Delta = \Delta(A_0, \dots, A_n)$. Choose the increasing sequence $(i_0, \dots, i_k)$ in $\{0, \dots, n\}$ such that for all $j = 0, ..., n$ there are nonnegative p_j, q_j such that $p_j/q_j \geq p_{j+1}/q_{j+1}$, with strict inequality for $j = i_l$ for some $l = 0, ..., k$. We put also $i_{-1} = -1$. Since P and Q are not contained in any proper face of Δ, we never have $p_j = q_j = 0$. Hence q_j/p_j is always a well defined number from $[0, +\infty]$.

Define $\underline{P} := \{P\}, \underline{Q} := \{Q\}, \underline{A}_j := \{A_l : i_{j-1} < l \leq i_j\} \in \mathrm{SET}(V), j = 0, \dots, k$. Consider 4 cases.

- $(0,0)$ $Q, P \in \mathrm{int}\,\Delta$.
- $(0,P)$ $Q \in \mathrm{int}\,\Delta, P \in \partial\Delta$.
- $(Q,0)$ $Q \in \partial\Delta, P \in \mathrm{int}\,\Delta$.
- (Q,P) $Q, P \in \partial\Delta$.

In the cases $(*,0)$, i.e. $P \in \mathrm{int}\,\Delta$, we have $\Delta_{(P)\,\mathrm{max}} = \{\Delta(A_0, \dots, \check{A}_i, \dots, A_n, P) : i = 0, \dots, n\}$. (Here $(*,0)$ means $(0,0)$ and $(Q,0)$.) Using our new notation we can write

$$\Delta_{(P)} = \langle\underline{A}_0, \dots, \underline{A}_k\rangle\,\{P\} \in \mathrm{SET}^2(V)$$

In the cases $(*,P)$, i.e. $P \in \partial\Delta$, we have $q_0/p_0 = ... = q_{i_0}/p_{i_0} = \infty$, and so $P \in \mathrm{int}\,\Delta(\underline{A}_1 \cup \dots \cup \underline{A}_k)$. Thus we have $\Delta_{(P)\,\mathrm{max}} = \{\Delta(A_0, \dots, \check{A}_i, \dots, A_n, P) :$

$i = i_0 + 1, \ldots, n\}$. We can write it as follows:

$$\Delta_{(P)} = \{\underline{A}_0\} \langle \underline{A}_1, \ldots, \underline{A}_k \rangle \{P\}$$

or

$$\Delta_{(P)} = \langle \underline{A}_0, \ldots, \underline{A}_k \rangle \{P\} \setminus \langle \underline{A}_0 \rangle \{\underline{A}_1 \cup \ldots \cup \underline{A}_k\} \in \mathrm{SET}^2(V).$$

Thus in all the cases we have:

$$\Delta_{(P)} = \langle \underline{A}_0, \ldots, \underline{A}_k \rangle \{P\} \setminus \delta^P \langle \underline{A}_0 \rangle \{\underline{A}_1 \cup \ldots \cup \underline{A}_k\} \in \mathrm{SET}^2(V),$$

where

$$\delta^P = \begin{cases} 0 & \text{in the cases } (*, 0), \\ 1 & \text{in the cases } (*, \ P). \end{cases}$$

In the cases $(0,*)$, i.e. $Q \in \mathrm{int}\,\Delta$, we have $Q \in \mathrm{int}\,\Delta(\underline{A}_0 \cup \ldots \underline{A}_{k-1} \cup \underline{P})$. Indeed

$$Q = \sum_{i=0}^n q_i A_i \ \text{ and } \ P = \sum_{i=0}^n p_i A_i.$$

Then

$$\begin{aligned} Q &= \sum_{i=0}^n q_i A_i + (q_{i_k}/p_{i_k})(P - \sum_{i=0}^n p_i A_i) \\ &= \sum_{i=0}^n (q_i - (q_{i_k}/p_{i_k})p_i) A_i + (q_{i_k}/p_{i_k})\, P. \end{aligned}$$

Here $q_i \neq 0$ since $Q \in \mathrm{int}\Delta$, $q_i \geq q_i - (q_{i_k}/p_{i_k})p_i \geq 0$; if $q_i/p_i = q_{i_k}/p_{i_k}$ then that coefficient is zero.

In the cases $(Q, *)$, i.e. $Q \in \partial\Delta$, we have $q_{i_{k-1}+1}/p_{i_{k-1}+1} = \ldots = q_{i_k}/p_{i_k} = 0$ and $Q \in \mathrm{int}\,\Delta(\underline{A}_0 \cup \ldots \cup \underline{A}_{k-1})$.

The triangulation $\Delta_{(P,Q)}$ is obtained by replacing in $\Delta_{(P)}$ all simplices containing Q by the convex hulls of Q and their faces which do not contain Q.

In all the considered cases the set of all maximal simplices of $\Delta_{(P)}$ containing Q is the following:

$$\{\underline{A}_0 \cup \ldots \cup \underline{A}_{k-1}\} \langle \underline{A}_k \rangle \{\underline{P}\}.$$

In the cases $(0,*)$ we replace this set by

$$\langle \underline{A}_0 \cup \ldots \cup \underline{A}_{k-1} \rangle \langle \underline{A}_k \rangle \{\underline{P}\} \{\underline{Q}\} \cup \{\underline{A}_0 \cup \ldots \cup \underline{A}_{k-1}\} \langle \underline{A}_k \rangle \{\underline{Q}\}.$$

In the cases $(Q, *)$ we replace it by

$$\langle \underline{A}_0 \cup \ldots \cup \underline{A}_{k-1} \rangle \langle \underline{A}_k \rangle \{\underline{P}\} \{\underline{Q}\}.$$

By the above

$$\begin{aligned} \Delta_{(P,Q)} = \ &\langle \underline{A}_0 \cup \ldots \cup \underline{A}_{k-1} \rangle \{\underline{A}_k\} \{\underline{P}\} \\ &\cup \langle \underline{A}_0 \cup \ldots \cup \underline{A}_{k-1} \rangle \langle \underline{A}_k \rangle \{\underline{P}\} \{\underline{Q}\} \\ &\cup \{\underline{A}_0 \cup \ldots \cup \underline{A}_{k-1}\} \langle \underline{A}_k \rangle \{\underline{Q}\} \\ &\setminus \delta^P \langle \underline{A}_0 \rangle \{\underline{A}_1 \cup \ldots \cup \underline{A}_k\} \{\underline{P}\} \setminus \delta^Q \{\underline{A}_0 \cup \ldots \cup \underline{A}_{k-1}\} \langle \underline{A}_k \rangle \{\underline{Q}\}, \end{aligned}$$

where

$$\delta^Q = \begin{cases} 0 & \text{in the cases } (0,*), \\ 1 & \text{in the cases } (Q, *). \end{cases}$$

By symmetry we have

$$\Delta_{(Q,P)} = \langle \underline{A}_0 \rangle \{ \underline{A}_1 \cup \ldots \cup \underline{A}_k \} \{ \underline{P} \}$$
$$\cup \langle \underline{A}_0 \rangle \langle \underline{A}_1 \cup \ldots \cup \underline{A}_k \rangle \{ \underline{P} \} \{Q\}$$
$$\cup \{ \underline{A}_0 \} \langle \underline{A}_1 \ldots \cup \underline{A}_k \rangle \{Q\}$$
$$\setminus \delta^P \langle \underline{A}_0 \rangle \{ \underline{A}_1 \cup \ldots \cup \underline{A}_k \} \{ \underline{P} \} \setminus \delta^Q \{ \underline{A}_0 \cup \ldots \cup \underline{A}_{k-1} \} \langle \underline{A}_k \rangle \{ \underline{Q} \}.$$

Lemma 4.1. *For $s = 0, \ldots, k-1$ define:*

$$\mathbf{T}_s := \langle \underline{A}_0 \cup \ldots \cup \underline{A}_s \rangle \{ \underline{A}_{s+1} \cup \ldots \cup \underline{A}_k \} \{ \underline{P} \}$$
$$\cup \langle \underline{A}_0 \cup \ldots \cup \underline{A}_s \rangle \langle \underline{A}_{s+1} \cup \ldots \cup \underline{A}_k \rangle \{ \underline{P} \} \{ \underline{Q} \}$$
$$\cup \{ \underline{A}_0 \cup \ldots \cup \underline{A}_s \} \langle \underline{A}_{s+1} \cup \ldots \cup \underline{A}_k \rangle \{ \underline{Q} \}$$
$$\setminus \delta^P \langle \underline{A}_0 \rangle \{ \underline{A}_1 \cup \ldots \cup \underline{A}_k \} \{ \underline{P} \} \setminus \delta^Q \{ \underline{A}_0 \cup \ldots \cup \underline{A}_{k-1} \} \langle \underline{A}_k \rangle \{ \underline{Q} \}.$$

Then $\mathbf{T}_s$ is a triangulation; $\mathbf{T}_0 = \Delta_{(Q,P)}$ and $\mathbf{T}_{k-1} = \Delta_{(P,Q)}$. Moreover $\mathbf{T}_s$ differs from $\mathbf{T}_{s+1}$ by an elementary transformation for $s = 0, \ldots, k-2$.

Proof. We use induction on s.

Assume that $\mathbf{T}_s$ is a triangulation. We have to show that $\mathbf{T}_{s+1}$ is a triangulation which differs from $\mathbf{T}_s$ by an elementary transformation.

Lemma 4.1.1. *For $s = 0, \ldots, k-1$ the set*

$$\mathbf{P^B}_s := \mathrm{conv}(\underline{A}_0 \cup \ldots \cup \underline{A}_s \cup \underline{P}; \underline{A}_{s+2} \cup \ldots \cup \underline{A}_k \cup \underline{Q})$$

is a basic polytope.

Proof. Recall that $Q = \sum_{i=0}^n q_i A_i$ and $P = \sum_{i=0}^n p_i A_i$; then

$$Q = \sum_{i=0}^n q_i A_i + (q_{i_{s+1}}/p_{i_{s+1}})(P - \sum_{i=0}^n p_i A_i)$$
$$= \sum_{i=0}^n (q_i - (q_{i_{s+1}}/p_{i_{s+1}})p_i)A_i + (q_{i_{s+1}}/p_{i_{s+1}})\, P.$$

Note that terms in i_{s+1} have coefficient 0. Hence we have

$$Q + \sum_{i=i_{s+1}+1}^n (q_{i_{s+1}}/p_{i_{s+1}} - q_i/p_i)p_i A_i$$
$$= (q_{i_{s+1}}/p_{i_{s+1}})P + \sum_{i=0}^{i_s}((q_i/p_i - q_{i_{s+1}}/p_{i_{s+1}})p_i A_i.$$

All coefficients in the above equality are positive. The sum of the LHS coefficients equals the sum of the RHS coefficients since we subtruct or add the same coefficients in both sides of the first equality. Therefore we can scale by that sum and get the relation in a basic polytope. On the other hand Q is independent of the set $(\underline{A}_0 \cup \ldots \cup \underline{A}_{s-1} \cup \underline{A}_{s+1} \cup \ldots \cup \underline{A}_k)$ since $q_i \neq 0$ for all $i \leq i_{k-1}$. Hence the above relation is the unique relation in the given polytope, which must be a basic polytope. ☐

Lemma 4.1.2. $\underline{P}_s^E = \mathrm{conv}(\{\underline{A}_0 \cup \ldots \cup \underline{A}_s \cup \underline{P}; \underline{A}_{s+2} \cup \ldots \cup \underline{A}_k \cup \underline{Q}\} : \langle \underline{A}_{s+1} \rangle)$ *for $s = 0, \ldots, k-2$ is a set of maximal elementary polytopes such that:*

(i) *All of the elementary polytopes have the same basic polytope $\mathbf{P^B}_s$.*

(ii) *For every polytope* $\mathbf{P}' = \mathrm{conv}(\underline{A}_0 \cup \ldots \cup \underline{A}_s \cup \underline{P}; \underline{A}_{s+2} \cup \ldots \underline{A}_k \cup \underline{Q} : *)$ *in* $\underline{P}_s^E$
    ($*$ *is an element of* $\langle A_{s+1}\rangle$) *the canonical triangulation*

$$\langle A_0 \cup \ldots \cup \underline{A}_s \cup \underline{P}\rangle\{\underline{A}_{s+2} \cup \ldots \cup \underline{A}_k \cup \underline{Q}\}\{*\}$$

    of $\mathbf{P}'$ *is a subcomplex of* $\mathbf{T}_s$.

(iii) *For any polytope* α *belonging to* $\mathrm{Star}(\{\underline{A}_{s+2} \cup \ldots \cup \underline{A}_k \cup \underline{Q}\} : \mathbf{T}_s\}$ *there exists*
    a polytope $\mathbf{P}' \in \underline{P}_s^E$ *such that* $\alpha \subset \mathbf{P}'$.

Proof. Let $\mathbf{P}' = \mathrm{conv}(\underline{A}_0 \cup \ldots \cup \underline{A}_s \cup \underline{P}; \underline{A}_{s+2} \cup \ldots \cup \underline{A}_k \cup \underline{Q} : *)$ be a polytope
in $\underline{P}_s^E$. Note that Q is not in the face $\mathrm{conv}(\underline{A}_0 \cup \ldots \cup \underline{A}_s \cup \underline{A}_{s+2} \cup \ldots \cup \underline{A}_k \cup *)$ of
Δ, since $q_i \neq 0$ for all $i_{s-1} + 1 \leq i \leq i_s$. Therefore Q is independent of the set of
vertices of $\mathrm{conv}(\underline{A}_0 \cup \ldots \cup \underline{A}_s \cup \underline{A}_{s+2} \cup \ldots \cup \underline{A}_k \cup *)$. Hence there is the only one
relation on the set of vertices of $\mathbf{P}'$. By the previous lemma $\mathbf{P}'$ is an elementary
polytope with the basic polytope $\mathbf{P}_s^{\mathbf{B}}$.
    (ii) follows from the fact that

$$\langle \underline{A}_0 \cup \ldots \cup \underline{A}_s \cup \underline{P}\rangle \{\underline{A}_{s+2} \cup \ldots \underline{A}_k \cup \underline{Q}\} \langle \underline{A}_{s+1}\rangle$$
$$= \{\underline{A}_0 \cup \ldots \cup \underline{A}_s\} \langle \underline{A}_{s+1}\rangle \{\underline{A}_{s+2} \cup \ldots \cup \underline{A}_k\} \{\underline{Q}\}$$
$$\cup\ \langle \underline{A}_0 \cup \ldots \cup \underline{A}_s\rangle \langle \underline{A}_{s+1}\rangle \{\underline{A}_{s+2} \cup \ldots \cup \underline{A}_k\} \{\underline{P}\} \{\underline{Q}\}$$

is a subset of $\mathbf{T}_s$.
    (iii) can be deduced from the fact that

$$\mathrm{Star}(\{\underline{A}_{s+2} \cup \ldots \cup \underline{A}_k \cup \underline{Q}\} ; \mathbf{T}_s)$$
$$= \langle \underline{A}_0 \cup \ldots \cup \underline{A}_s \cup \underline{P}\rangle \langle \underline{A}_{s+1}\rangle \{\underline{A}_{s+2} \cup \ldots \cup \underline{A}_k\} \{\underline{Q}\}$$
$$\subset\ \mathrm{conv}(\underline{A}_0 \cup \ldots \cup \underline{A}_s \cup \underline{A}_{s+2} \cup \ldots \cup \underline{A}_k \cup \underline{P} \cup \underline{Q})\langle \underline{A}_{s+1}\rangle. \quad \square$$

By Lemmas 4.0 and 4.1.2 and since $\mathbf{T}_s$ is a triangulation, we get that the set
$\underline{P}_s^E$ determines an elementary transformation. It changes the set

$$\langle \underline{A}_0 \cup \ldots \cup \underline{A}_s \cup \underline{P}\rangle \{\underline{A}_{s+2}\cup \ldots \cup\underline{A}_k \cup \underline{Q}\} \langle \underline{A}_{s+1}\rangle \subset \mathbf{T}_s$$

into a set $\{\underline{A}_0\cup \ldots \cup\underline{A}_s \cup \underline{P}\} \langle \underline{A}_{s+2}\cup \ldots \cup\underline{A}_k \cup \underline{Q}\rangle \langle \underline{A}_{s+1}\rangle \subset \mathbf{T}_{s+1}$, and transforms
the triangulation $\mathbf{T}_s$ into $\mathbf{T}_{s+1}$. In particular the set $\mathbf{T}_{s+1}$ is a triangulation.
    Lemma 4.1 is proved. $\square$

To complete the proof of Proposition 4A, note that all the considered triangula-
tions restricted to $\partial\Delta$ are equal to $\partial\Delta_{(P,Q)}$. $\square$

Corollary 4.1.3. *Each basic polytope in the above elementary equivalence is of the
form*

$$(\mathbf{P}^{\mathbf{B}})' = \mathrm{conv}(A_{i_1}, \ldots, A_{i_h}, P; A_{i_{h+1}}, \ldots, A_{i_g}, Q)$$

for some subset $\{i_1, \ldots, i_g\} \subset \{1, \ldots, n\}$ *where* $0 \leq h < g$. $\square$

Proof of Proposition 4B. Pair $(\mathbf{P}^{\mathbf{B}}; \{Q\})$, where $\mathbf{P}^{\mathbf{B}}$ is a basic polytope and $Q \in$
$\mathbf{P}^{\mathbf{B}}$.

Consider two projections resp. on the affine space $\mathrm{aff}(A_0, \ldots, A_n)$ in direction
$\mathrm{aff}(B_0, \ldots, B_m)$ and vice versa. Let r_0, r_1 be their restriction to $\mathbf{P}^{\mathbf{B}}$. We have

$$r_0 : \mathbf{P}^{\mathbf{B}} \longrightarrow \Delta(A_0, \ldots, A_n),$$
$$r_1 : \mathbf{P}^{\mathbf{B}} \longrightarrow \Delta(B_0, \ldots, B_m),$$

$$r_0(\sum a_i A_i + \sum b_j B_j) = \sum a_i A_i + (\sum b_i)P,$$
$$r_i(\sum a_i A_i + \sum b_j B_j) = (\sum a_i)P + \sum b_i B_i,$$

where $P := \Delta(A_0, \ldots, A_n) \cap \Delta(B_0, \ldots, B_m)$.

Denote $r_0(Q)$ by Q_0, and $r_1(Q)$ by Q_1. For the pairs $(\Delta(A_0, \ldots, A_n) : \{Q_0, P\})$; $(\Delta(B_0, \ldots, B_m) : \{Q_i, P\})$ choose notations as in the previous case: $\underline{A} := \{A_0, \ldots, A_n\} = \underline{A}_0 \cup \ldots \cup \underline{A}_l$ and $\underline{B} := \{B_0, \ldots, B_m\} = \underline{B}_0 \cup \ldots \cup \underline{B}_d$. Set $\underline{Q} := \{Q\}$; and $\underline{P} := \{P\}$

We shall represent triangulations of the pair $(\mathbf{P^B}; P)$ as elements of $SET^2(V)$, where $V = \{A_0, \ldots, A_n, B_0, \ldots, B_m, Q\}$. For instance $\mathbf{B} = \langle \underline{A} \rangle \{\underline{B}\}$.

If $P = Q$ then $\mathbf{A}_{(Q)} = \mathbf{B}_{(Q)} = \langle \underline{A} \rangle \langle \underline{B} \rangle \{\underline{Q}\}$, and we are done. Assume $P \neq Q$. Consider the following cases.

$\quad$ 1. $\quad Q \notin \partial \mathbf{P^B}:$ $\quad (a)$ $\; Q \notin \Delta(B_0, \ldots, B_m)$,

$\qquad\qquad\qquad\qquad\quad (b)$ $\; Q \in \mathrm{int}\, \Delta(B_0, \ldots, B_m)$;

$\quad$ 2. $\quad Q \in \partial \mathbf{P^B}:$ $\quad (a)$ $\; Q \in \partial \Delta(B_0, \ldots, B_m)$,

$\qquad\qquad\qquad\qquad\quad (b)$ $\; Q \in \partial \Delta(A_0, \ldots, A_n)$,

$\qquad\qquad\qquad\qquad\quad (c)$ $\; Q \in \partial \mathbf{P^B} \setminus \partial \Delta(B_0, \ldots, B_m) \setminus \partial \Delta(A_0, \ldots, A_n)$.

Case 1a. We then have $Q \in \mathrm{int}\, \Delta(B_0, \ldots, B_m, A_{i_0}, \ldots, A_{i_p})$ for some $i_0, \ldots, i_p$. Then $Q_0 \in \mathrm{int}\, \Delta(P, A_{i_0}, \ldots, A_{i_p}) = \mathrm{int}\, \Delta(\underline{P}, \underline{A}_0 \cup \ldots \cup \underline{A}_{l-1})$ (as in the $(0,*)$ case). Hence $\{A_{i_0}, \ldots, A_{i_p}\} = \underline{A}_0 \cup \ldots \cup \underline{A}_{l-1}$. Finally, $Q \in \mathrm{int}\, \Delta(\underline{B}, \underline{A}_0 \cup \ldots \cup \underline{A}_{l-1})$. We obtain $\mathbf{B}_{(Q)}$ from $\mathbf{B} = \langle \underline{A} \rangle \{\underline{B}\}$ by replacing

$$\{\underline{B} \cup \underline{A}_0 \cup \ldots \cup \underline{A}_{l-1}\} \langle \underline{A}_l \rangle$$

with

$$\langle \underline{A}_0 \cup \ldots \cup \underline{A}_{l-1} \cup \underline{B} \rangle \langle \underline{A}_l \rangle \{Q\}$$
$$= \langle \underline{A}_0 \cup \ldots \cup \underline{A}_{l-1} \rangle \langle \underline{A}_l \rangle \{\underline{B}\} \{\underline{Q}\}$$
$$\cup \; \{\underline{A}_0 \cup \ldots \cup \underline{A}_{l-1}\} \langle \underline{A}_l \rangle \langle \underline{B} \rangle \{\underline{Q}\}.$$

Hence

$$\mathbf{B}_{(Q)} \quad = \langle \underline{A}_0 \cup \ldots \cup \underline{A}_{l-1} \rangle \{\underline{A}_l\} \{\underline{B}\} \; \cup \; \langle \underline{A}_0 \cup \ldots \cup \underline{A}_{l-1} \rangle \langle \underline{A}_l \rangle \{\underline{B}\} \{\underline{Q}\}$$
$$\cup \; \{\underline{A}_0 \cup \ldots \cup \underline{A}_{l-1}\} \langle \underline{A}_l \rangle \langle \underline{B} \rangle \{\underline{Q}\}.$$

Since $Q \in \mathrm{int}\, \mathbf{P^B}$ we have $\mathbf{P^B}_{(Q)} = \langle \underline{A} \rangle \langle \underline{B} \rangle \{Q\}$.

Case 1b. $\mathbf{B}_{(Q)}$ is obtained from $\mathbf{B}$ by changing all its simplices into the set of simplices $\langle \underline{A} \rangle \langle \underline{B} \rangle \{Q\}$. Then $\mathbf{P^B}_{(Q)} = \langle \underline{A} \rangle \langle \underline{B} \rangle \{Q\}$, as in the previous case.

Case 2a. $Q = \overline{Q}_1 \in \mathrm{int}\, \Delta(\underline{B}_0 \cup \ldots \cup \underline{B}_{d-1})$. We insert the set of all simplices from $\langle \underline{A} \rangle \{\underline{B}\}$ into the set $\mathbf{B}_{(Q)} = \langle \underline{A} \rangle \langle \underline{B}_0 \cup \ldots \cup \underline{B}_{d-1} \rangle \{\underline{B}_d\} \{Q\}$.

Note also that $\mathbf{P^B}_{(Q)} = \mathbf{B}_{(Q)} = \langle \underline{A} \rangle \langle \underline{B}_0 \cup \ldots \cup \underline{B}_{d-1} \rangle \{\underline{B}_d\} \{Q\}$.

Case 2b. $Q = Q_0 \in \mathrm{int}\, \Delta(\underline{A}_0 \cup \ldots \cup \underline{A}_{l-1})$. We replace $\{\underline{A}_0 \cup \ldots \cup \underline{A}_{l-1}\} \langle \underline{A}_l \rangle \{\underline{B}\}$ by $\langle \underline{A}_0 \cup \ldots \cup \underline{A}_{l-1} \rangle \langle \underline{A}_l \rangle \{\underline{B}\} \{\underline{Q}\}$ and get the triangulation
$\mathbf{B}_{(Q)} = \langle \underline{A}_0 \cup \ldots \cup \underline{A}_{l-1} \rangle \langle \underline{A}_l \rangle \{\underline{B}\} \{\underline{Q}\} \cup \langle \underline{A}_0 \cup \ldots \cup \underline{A}_{l-1} \rangle \{\underline{A}_l\} \{\underline{B}\}$,

$$\mathbf{P^B}_{(Q)} = \langle \underline{A}_0 \cup \ldots \cup \underline{A}_{l-1} \rangle \{\underline{A}_l\} \langle \underline{B} \rangle \{Q\}.$$

Case 2c. By taking projections r_0, r_1 we get that

$$Q \in \mathrm{int}\, \Delta\big(\underline{A}_0 \cup \ldots \cup \underline{A}_{l-1} \cup \underline{B}_0 \cup \ldots \cup \underline{B}_{d-1}\big).$$

We replace the subsets $\{\underline{A}_0\cup\ldots\cup\underline{A}_{l-1}\cup\underline{B}\}\langle\underline{A}_l\rangle$ of $\langle\underline{A}\rangle\{B\}$ with the set

$$\langle\underline{A}_0\cup\ldots\cup\underline{A}_{l-1}\cup\underline{B}_0\cup\ldots\cup\underline{B}_{d-1}\rangle\langle\underline{A}_l\rangle\{\underline{B}_d\}\,\{\underline{Q}\}$$
$$=\langle\underline{A}_0\cup\ldots\cup\underline{A}_{l-1}\rangle\langle\underline{A}_l\rangle\,\{\underline{B}\}\,\{\underline{Q}\}$$
$$\cup\,\{\underline{A}_0\cup\ldots\cup\underline{A}_{l-1}\}\,\langle\underline{A}_l\rangle\,\langle\underline{B}_0\cup\ldots\cup\underline{B}_{d-1}\rangle\,\{\underline{B}_d\}\,\{\underline{Q}\}$$

and get the triangulation

$$\mathbf{B}_{(Q)} =\langle\underline{A}_0\cup\ldots\cup\underline{A}_{l-1}\rangle\,\{\underline{A}_l\}\,\{\underline{B}\}$$
$$\cup\,\langle\underline{A}_0\cup\ldots\cup\underline{A}_{l-1}\rangle\,\langle\underline{A}_l\rangle\,\{\underline{B}\}\,\{\underline{Q}\}$$
$$\cup\,\{\underline{A}_0\cup\ldots\cup\underline{A}_{l-1}\}\,\langle\underline{A}_l\rangle\,\langle\underline{B}_0\cup\ldots\cup\underline{B}_{d-1}\rangle\,\{\underline{B}_d\}\,\{\underline{Q}\},$$

$$\mathbf{P^B}_{(Q)} = \langle\underline{A}\rangle\,\langle\underline{B}\rangle\,\{\underline{Q}\}\setminus\{\underline{A}_0\cup\ldots\cup\underline{A}_{l-1}\}\{\underline{B}_0\cup\ldots\cup\underline{B}_{d-1}\}\langle\underline{A}_l\rangle\langle\underline{B}_d\rangle\{Q\}$$

Note that in cases 1b and 2a $\mathbf{P^B}_{(Q)} = \mathbf{B}_{(Q)}$. Consider the other cases.
Let $\mathbf{T}_s$ denotes the following triangulations:

$$\mathbf{T}_s :=\langle\underline{A}_0\cup\ldots\cup\underline{A}_{l-s}\rangle\,\{\underline{A}_{l-s+1}\cup\ldots\cup\underline{A}_s\}\,\{\underline{B}\}$$
$$\cup\,\langle\underline{A}_0\cup\ldots\cup\underline{A}_{l-s}\rangle\,\langle\underline{A}_{l-s+1}\cup\ldots\cup\underline{A}_s\rangle\,\{\underline{B}\}\,\{\underline{Q}\}$$
$$\cup\,\{\underline{A}_0\cup\ldots\cup\underline{A}_{l-s}\}\langle\underline{A}_{l-s+1}\cup\ldots\cup\underline{A}_s\rangle\langle\underline{B}\rangle\,\{\underline{Q}\}\setminus\,S,$$

where

$$S = \begin{cases} \emptyset \text{ in case 1a,} \\ \{\underline{A}_0\cup\ldots\cup\underline{A}_{l-1}\}\,\langle\underline{A}_l\rangle\,\langle\underline{B}\rangle\,\{\underline{Q}\}\ \text{in case 2b,} \\ \{\underline{A}_0\cup\ldots\cup\underline{A}_{l-1}\}\,\langle\underline{A}_l\rangle\,\{\underline{B}_0\cup\ldots\cup\underline{B}_{d-1}\}\,\langle\underline{B}_d\rangle\,\{\underline{Q}\}\ \text{in case 2c.} \end{cases}$$

Lemma 4.2.   (a) $\mathbf{T}_1 = B_{(Q)}$.
(b) $\mathbf{T}_s$ *is a triangulation for* $s = 1,...,l$.
(c) $\mathbf{T}_s$ *is el-equivalent to* $\mathbf{T}_{s-1}$ *for* $s = 2,\ldots,l$.
(d) $\mathbf{T}_l$ *is el-equivalent to* $\mathbf{P^B}_{(Q)}$.

It follows from the lemma that $\mathbf{B}_{(Q)}$ is el-equivalent to $\mathbf{P^B}_{(Q)}$. By symmetry we have that $\mathbf{A}_{(Q)}$ is el-equivalent to $\mathbf{P^B}_{(Q)}$. Finally the lemma yields the el-equivalence of $\mathbf{A}_{(Q)}$ and $\mathbf{B}_{(Q)}$.

Proof. (a) follows just from the definition of $\mathbf{T}_s$. In case 2c by properties 1,2,3 of our operations we have

$$\{\underline{A}_0\cup\ldots\cup\underline{A}_{l-1}\}\,\langle\underline{A}_l\rangle\langle\underline{B}_0\cup\ldots\cup\underline{B}_{d-1}\rangle\,\{\underline{B}_d\}\,\{\underline{Q}\}$$
$$= \{\underline{A}_0\cup\ldots\cup\underline{A}_{l-1}\}\langle\underline{A}_l\rangle\langle\underline{B}\rangle\,\{\underline{Q}\}$$
$$\setminus\{A_0\cup\ldots\cup\underline{A}_{l-1}\}\,\langle\underline{A}_l\rangle\,\{\underline{B}_0\cup\ldots\cup\underline{B}_{d-1}\}\,\langle\underline{B}_d\rangle\,\{\underline{Q}\}.$$

Other cases are tautological. We prove (b) and (c) by induction on s.

Lemma 4.2.1. *For* $s = 1,...,l-1$

$$\mathbf{P^B}_s := \mathrm{conv}(\underline{A}_0\cup\ldots\cup\underline{A}_{l-s-1}\cup\underline{B}\,;\,\underline{A}_{l-s+1}\cup\ldots\cup\underline{A}_l\cup\underline{Q})$$

is a basic polytope.

Proof. By Lemma 4.1.1 $\mathrm{conv}(\underline{A}_0\cup\ldots\cup\underline{A}_{l-s-1}\cup\underline{P}\,;\,\underline{A}_{l-s+1}\cup\ldots\cup\underline{A}_l\cup\underline{Q}_0)$ is a basic polytope. There exist $a_j, p, q > 0$ such that $\sum_{j\le i_{l-s-1}} a_j + p = \sum_{j>i_{l-s}} a_j + q = 1$ and

$$(0)\qquad\qquad \sum_{j\le i_{l-s-1}} a_j A_j + pP = \sum_{j>i_{l-s}} a_j A_j + qQ_0.$$

By the definition of P

$$\text{(1)} \qquad P = \sum_{i=0}^{n} p_i B_i, \quad \text{where } p_i > 0 \text{ and } \sum_{i=0}^{n} p_i = 1.$$

There exist a'_j and b'_i such that $Q = \sum a'_i A_i + \sum b'_j B_j$. Then by definition of Q_0 and r_0

$$Q_0 = \sum a'_i A_i + (\sum b'_j) P = Q - \sum b'_j B_j + (\sum b'_j) P = Q - (\sum (b'_j - (\sum b'_j) p_j) B_j),$$

$$\text{(2)} \qquad Q_0 = Q + \sum k_i B_i, \text{ where } \sum k_i = 0.$$

Putting (1) and (2) into (0) we get :

$$\text{(3)} \qquad \sum_{j \leq i_{l-s-1}} a_j A_j + \sum_{i \in I} b_i B_i = \sum_{j > i_{l-s}} a_j A_j + qQ + \sum_{i \in I'} b_i B_i,$$

where $b_i = |pp_i - qk_i|$ and $pp_i - qk_i \geq 0$ iff $i \in I$. In particular all the coefficients are non-negative, and I and I' denote two disjoint sets of indices such that $I \cup I' \subseteq \{0, \ldots, m\}$.

The set I is non-empty since some k_i in (2) are non-positive. Assume that I is not a proper subset of $\{0, \ldots, m\}$, i.e. I' is nonempty. Then the right side of (3) would be a point of the face $\mathbf{P}' = \text{conv}(A_j, B_i : j \leq i_{l-s-1}, i \in I)$ of $\mathbf{P}^B$. If $Q \notin \mathbf{P}'$ then by (3) we get that the all points on RHS $A_j, j \geq i_{l-s}, B_i, i \in I'$, do not belong to the face $\mathbf{P}'$ of $\mathbf{P}^B$. Since these points belong to $\mathbf{P}^B$ and do not belong to $\mathbf{P}'$, their convex hull does not intersect $\mathbf{P}'$, which contradicts (3). If $Q \in \mathbf{P}'$ then $Q_0 \in \text{conv}(A_j, P : j \leq i_{l-s-1})$, which is not possible since $\text{conv}(\underline{A}_0 \cup \ldots \cup \underline{A}_{l-s-1} \cup \underline{P} : \underline{A}_{l-s+1} \cup \ldots \cup \underline{A}_l \cup \underline{Q}_0)$ is a basic polytope. Finally $I = \{0, \ldots, m\}$ and I' is empty. We can write (3) as

$$\text{(4)} \qquad \sum_{j \leq i_{l-s-1}} a_j A_j + \sum b_i B_i = \sum_{j > i_{l-s}} a_j A_j + qQ.$$

Note also that (4) is the only relation on the set of vertices. $\qquad \square$

Lemma 4.2.2. $\underline{P}_s^E = \text{conv}(\{\underline{A}_0 \cup \ldots \cup \underline{A}_{l-s-1} \cup \underline{B} ; A_{l-s+1} \cup \ldots \cup \underline{A}_l \cup \underline{Q}\}\langle \underline{A}_{l-1}\rangle)$ *is a set of maximal elementary polytopes such that:*

1. *All of the elementary polytopes have the same basic polytope $\mathbf{P^B}_s$.*
2. *For every polytope $\mathbf{P}' = \text{conv}(\underline{A}_0 \cup \ldots \cup \underline{A}_{l-s-1} \cup \underline{B} ; \underline{A}_{l-s-1} \cup \ldots \cup \underline{A}_l \cup \underline{Q}; *) \in \underline{P}_s^E$ the canonical triangulation*

$$\{\underline{A}_0 \cup \ldots \cup \underline{A}_{l-s-1} \cup \underline{B}\}\langle \underline{A}_{l-s+1} \cup \ldots \cup \underline{A}_l \cup \underline{Q}\rangle \{*\}$$

   of $\mathbf{P}'$ is a subcomplex of $\mathbf{T}_s$.
3. *For every polytope α belonging to $\text{Star}(\{\underline{A}_0 \cup \ldots \cup \underline{A}_{l-s-1} \cup \underline{B}\} : \mathbf{T}_s)$ there exists a polytope $\mathbf{P}' \in \underline{P}_s^E$ such that $\alpha \subset \mathbf{P}'$.*

Proof. As for Lemma 4.1.2.

By Lemmas 4.0 and 4.2.2, and since $\mathbf{T}_s$ is a triangulation, the set $\underline{P}_s^E$ determines an elementary transformation. It changes the set

$$\{\underline{A}_0 \cup \ldots \cup \underline{A}_{l-s-1} \cup \underline{B}\}\langle \underline{A}_{l-s+1} \cup \ldots \cup \underline{A}_l \cup \underline{Q}\rangle\langle \underline{A}_{l-s}\rangle \subseteq \mathbf{T}_s$$

into

$$\langle \underline{A}_0 \cup \ldots \cup \underline{A}_{l-s-1} \cup \underline{B}\rangle\{\underline{A}_{l-s+1} \cup \ldots \cup \underline{A}_l \cup \underline{Q}\}\langle \underline{A}_{l-s}\rangle \subseteq \mathbf{T}_{s+1},$$

transforming $\mathbf{T}_s$ to the triangulation $\mathbf{T}_{s+1}$.   □

The elementary transformation with the basic complex

$$\operatorname{conv}(\underline{B} : \underline{Q} \cup \underline{A}_1 \cup \ldots \cup \underline{A}_l)\langle\underline{A}_0\rangle$$

changes the set

$$\langle\underline{A}_0\rangle \ \langle A_1 \cup \ldots \cup \underline{A}_l\rangle \ \{\underline{B}\} \ \{\underline{Q}\} \cup \{\underline{A}_1 \cup \ldots \cup \underline{A}_l\} \ \{\underline{B}\} \subseteq \mathbf{T}_l$$

into

$$\langle\underline{A}_0\rangle \ \{\underline{A}_1 \cup \ldots \cup \underline{A}_l\} \ \langle\underline{B}\rangle \ \{\underline{Q}\} \ \subseteq \ \mathbf{P^B}_{(Q)}$$

and transforms $\mathbf{T}_l$ to $\mathbf{P}^B_{(Q)}$. Lemma 4.2 is proved.   □

To complete the proof of Proposition 4B, note that all the considered triangulations restricted to the boundary $\partial\mathbf{P}^B$ of $\mathbf{P}^B$ are equal to $(\partial\mathbf{P}^B)_{(Q)}$.   ∎

Corollary 4.2.3. *Let $Q \in \operatorname{int}\Delta(A_0, \ldots, A_n)$. Then $\mathbf{B}_{(Q)}$ and $\mathbf{A}_{(Q)}$ differ by a succesion of elementary transformations with basic polytopes of the form*

$$\operatorname{conv}(A_{i_1}, \ldots A_{i_h}, Q; B_0, \ldots B_m, A_{i_{h+1}}, \ldots, A_{i_g})$$

for some subset $\{i_1, \ldots, i_g\} \subset \{0, \ldots, n\}$ where $h \leq g$

Proof. It follows from the previous proof that $\mathbf{B}_{(Q)}$ and $\mathbf{P^B}_{(Q)}$ differ by elementary transformations with basic polytopes of the form

$$\operatorname{conv}(A_{i_1}, \ldots, A_{i_h}, Q; B_0, \ldots, B_m, A_{i_{h+1}}, \ldots, A_{i_g})$$

where $h \leq g$.

In case 1a we have that $\mathbf{P^B}_{(Q)} = \mathbf{B}_{(Q)}$. The case $Q \in \operatorname{int}\Delta(A_0, \ldots, A_n)$ is symmetric to case 1a. Thus we have $\mathbf{P^B}_{(Q)} = \mathbf{A}_{(Q)}$.

Finally, $\mathbf{B}_{(Q)}$ and $\mathbf{A}_{(Q)}$ differ by elementary transformations with basic polytopes of the given form.   ∎

Proof of Proposition 4C. Consider a pair $(\mathbf{P^E}; Q)$, where $\mathbf{P^E}, Q$ are as in part $\mathbf{C}$ of Proposition 4.

Let $\underline{C} = \{C_1, \ldots, C_r\}$, $\underline{A} = \{A_0, \ldots, A_n\}$, $\underline{B} = \{B_0, \ldots, B_m\}$, $\underline{Q} = \{Q\}$.

Assume that $Q \in \Delta(C_1, \ldots, C_r)$. Thus $\mathbf{B}_{(Q)} = \langle\underline{A}\rangle \ \{\underline{B}\} \ \langle\underline{C}\rangle \ \{\underline{Q}\}$, $\mathbf{A}_{(Q)} = \{\underline{A}\} \ \langle\underline{B}\rangle \ \langle\underline{C}\rangle \ \{\underline{Q}\}$, and $\mathbf{A}_{(Q)}$ differs from $\mathbf{B}_{(Q)}$ by an elementary transformation on the basic complex $\{\underline{A} \cup \underline{B}\}\langle\underline{C}\rangle\{\underline{Q}\}$; and we are done.

Thus now we assume that $Q \notin \Delta(C_1, \ldots, C_r)$. Let p be the projection from the face $\Delta(C_1, \ldots, C_r)$ of $\mathbf{P^E}$ onto the basic polytope $\mathbf{P^B} = \operatorname{conv}(A_0, \ldots, A_n : B_0, \ldots, B_m)$. Write in affine coordinates

$$p : \mathbf{P^E} \setminus \Delta(C_1, \ldots, C_P) \longrightarrow \mathbf{P^B},$$

$$p(\sum a_i A_i + \sum b_j B_j + \sum c_k C_k) = \sum (a_i/c) \ A_i + \sum (b_j/c) \ B_j,$$

where $c = 1 - \sum c_k > 0$.

Let $Q' = p(Q)$. For $Q' \in \mathbf{P}^B$ find the sets $\underline{A}_i$ and $\underline{B}_j$ as in the case 4B. This allows us to represent triangulations of the considered pair as elements of $\operatorname{SET}^2(V)$ where $V = \{A_0, \ldots, A_n, B_0, \ldots, B_m, C_1, \ldots, C_r, Q\}$. (See 4A and 4B.)

We represent the triangulation $\mathbf{B}_{(Q)}$. We have $\mathbf{B} = \langle \underline{A} \rangle \cdot \{\underline{B}\} \cdot \{\underline{C}\}$. Consider several cases.

1. $Q' \notin \partial \mathbf{P}^{\mathbf{B}} :$ (a) $Q' \notin \Delta(B_0, \dots, B_m),$
                                                      (b) $Q' \in \mathrm{int}\, \Delta(B_0, \dots, B_m);$
2. $Q' \in \partial \mathbf{P}^{\mathbf{B}} :$ (a) $Q' \in \partial \Delta(B_0, \dots, B_m),$
                                                     (b) $Q' \in \partial \Delta(A_0, \dots, A_n),$
                                                     (c) $Q' \in \partial \mathbf{P}^{\mathbf{B}} \setminus \partial \Delta(B_0, \dots, B_m) \setminus \partial \Delta(A_0, \dots, A_n).$

Case 1a. As before, $Q' \in \mathrm{int}\, \Delta(\underline{B}, \underline{A}_0, \cup \dots \cup \underline{A}_{l-1})$; hence

$$Q \in \mathrm{int}\, \Delta(\underline{B}, \underline{A} \cup \dots \cup \underline{A}_{l-1}, \underline{C}).$$

$\mathbf{B}_{(Q)}$ is obtained from $\mathbf{B} = \langle \underline{A} \rangle \{\underline{B}\} \{\underline{C}\}$ by changing $\{\underline{B} \cup \underline{A}_0 \cup \dots \cup \underline{A}_{l-1} \cup \underline{C}\} \langle \underline{A}_l \rangle$ into

$$\langle \underline{A}_0 \cup \dots \cup \underline{A}_{l-1} \cup \underline{B} \cup \underline{C} \rangle \langle \underline{A}_l \rangle \{Q\}$$
$$= \langle \underline{A}_0 \cup \dots \cup \underline{A}_{l-1} \rangle \langle A_l \rangle \{\underline{B}\} \{\underline{C}\} \{\underline{Q}\}$$
$$\cup \{\underline{A}_0 \cup \dots \cup \underline{A}_{l-1}\} \langle A_l \rangle \langle \underline{B} \rangle \{\underline{C}\} \{\underline{Q}\}$$
$$\cup \{\underline{A}_0 \cup \dots \cup \underline{A}_{l-1}\} \langle A_l \rangle \{\underline{B}\} \langle \underline{C} \rangle \{\underline{Q}\}.$$

Hence:

$$\mathbf{B}_{(Q)} = \langle \underline{A}_0 \cup \dots \cup \underline{A}_{l-1} \rangle \{\underline{A}_l\} \{\underline{B}\} \{\underline{C}\}$$
$$\cup \langle \underline{A}_0 \cup \dots \cup \underline{A}_{l-1} \rangle \langle A_l \rangle \{\underline{B}\} \{\underline{C}\} \{\underline{Q}\}$$
$$\cup \{\underline{A}_0 \cup \dots \cup \underline{A}_{l-1}\} \langle A_l \rangle \langle \underline{B} \rangle \{\underline{C}\} \{\underline{Q}\}$$
$$\cup \{\underline{A}_0 \cup \dots \cup \underline{A}_{l-1}\} \langle A_l \rangle \{\underline{B}\} \langle \underline{C} \rangle \{\underline{Q}\}.$$

Case 1b. $Q' \in \mathrm{int}\, \Delta(\underline{B}) \implies Q \in \mathrm{int}\, \Delta(\underline{B} \cup \underline{C}))$. $\mathbf{B}_{(Q)}$ is obtained from $\mathbf{B}$ by changing the set of all its simplices into the set

$$\mathbf{B}_{(Q)} = \langle \underline{A} \rangle \langle \underline{B} \rangle \{\underline{C}\} \{\underline{Q}\} \cup \langle \underline{A} \rangle \{\underline{B}\} \langle \underline{C} \rangle \{\underline{Q}\}.$$

Case 2a. $Q' \in \mathrm{int}\, \Delta(\underline{B}_0 \cup \dots \cup \underline{B}_{d-1})$. Hence $Q \in \mathrm{int}\, \Delta(\underline{B}_0 \cup \dots \cup \underline{B}_{d-1} \cup \underline{C})$. We replace the set of all simplices $\langle \underline{A} \rangle \{\underline{B}\} \{\underline{C}\}$ by the set

$$\mathbf{B}_{(Q)} = \langle \underline{A} \rangle \langle \underline{B}_0 \cup \dots \cup \underline{B}_{d-1} \rangle \{\underline{B}_d\} \{\underline{C}\} \{Q\} \cup \langle \underline{A} \rangle \{\underline{B}\} \langle \underline{C} \rangle \{\underline{Q}\}.$$

Case 2b. Then $Q \in \mathrm{int}\, \Delta(\underline{A}_0 \cup \dots \cup \underline{A}_{l-1} \cup \underline{C})$. We replace

$$\{A_0 \cup \dots \cup \underline{A}_{l-1}\} \langle \underline{A}_l \rangle \{\underline{B}\} \{\underline{C}\} \subset \langle \underline{A} \rangle \{\underline{B}\} \{\underline{C}\}$$

by

$$\langle \underline{A}_0 \cup \dots \cup \underline{A}_{l-1} \rangle \langle \underline{A}_l \rangle \{\underline{B}\} \{\underline{C}\} \{\underline{Q}\}$$
$$\cup \{\underline{A}_0 \cup \dots \cup \underline{A}_{l-1}\} \langle \underline{A}_l \rangle \{\underline{B}\} \langle \underline{C} \rangle \{\underline{Q}\}$$

to get

$$\mathbf{B}_{(Q)} = \langle \underline{A}_0 \cup \dots \cup \underline{A}_{l-1} \rangle \{\underline{A}_l\} \{\underline{B}\} \{\underline{C}\}$$
$$\cup \langle \underline{A}_0 \cup \dots \cup \underline{A}_{l-1} \rangle \langle \underline{A}_l \rangle \{\underline{B}\} \{\underline{C}\} \{\underline{Q}\}$$
$$\cup \{\underline{A}_0 \cup \dots \cup \underline{A}_{l-1}\} \langle \underline{A}_l \rangle \{\underline{B}\} \langle \underline{C} \rangle \{\underline{Q}\}.$$

Case 2c. Then $Q' \in \mathrm{int}\, \Delta(\underline{A}_0 \cup \dots \cup \underline{A}_{l-1} \cup \underline{B}_0 \cup \dots \cup \underline{B}_{d-1})$. Hence

$$Q \in \mathrm{int}\, \Delta(\underline{A}_0 \cup \dots \cup \underline{A}_{l-1} \cup \underline{B}_0 \cup \dots \cup \underline{B}_{d-1} \cup \underline{C}).$$

We replace the set $\{\underline{A}_0 \cup \ldots \cup \underline{A}_{l-1} \cup \underline{B} \cup \underline{C}\} \langle \underline{A}_l \rangle$ by

$$\langle \underline{A}_0 \cup \ldots \cup \underline{A}_{l-1} \cup \underline{B}_0 \cup \ldots \cup \underline{B}_{d-1} \cup \underline{C} \rangle \, \langle \underline{A}_l \rangle \, \{\underline{B}_d\} \, \{\underline{Q}\}$$

$$= \langle \underline{A}_0 \cup \ldots \cup \underline{A}_{l-1} \rangle \, \langle \underline{A}_l \rangle \, \{\underline{B}\} \, \{\underline{C}\} \, \{\underline{Q}\}$$

$$\cup \, \{\underline{A}_0 \cup \ldots \cup \underline{A}_{l-1}\} \, \langle \underline{A}_l \rangle \, \{\underline{B}\} \, \langle \underline{C} \rangle \, \{\underline{Q}\}$$

$$\cup \, \{\underline{A}_0 \cup \ldots \cup \underline{A}_{l-1}\} \, \langle \underline{A}_l \rangle \, \langle \underline{B}_0 \cup \ldots \cup \underline{B}_{d-1} \rangle \, \{\underline{B}_d\} \, \{\underline{C}\} \, \{\underline{Q}\}$$

to get

$$\mathbf{B}_{(Q)} = \langle \underline{A}_0 \cup \ldots \cup \underline{A}_{l-1} \rangle \, \{\underline{A}_l\} \, \{\underline{B}\} \, \{\underline{C}\}$$

$$\cup \, \langle \underline{A}_0 \cup \ldots \cup \underline{A}_{l-1} \rangle \, \langle \underline{A}_l \rangle \, \{\underline{B}\} \, \{\underline{C}\} \, \{\underline{Q}\}$$

$$\cup \, \{\underline{A}_0 \cup \ldots \cup \underline{A}_{l-1}\} \, \langle \underline{A}_l \rangle \, \{\underline{B}\} \, \langle \underline{C} \rangle \, \{\underline{Q}\}$$

$$\cup \, \{\underline{A}_0 \cup \ldots \cup \underline{A}_{l-1}\} \, \langle \underline{A}_l \rangle \, \langle \underline{B}_0 \cup \ldots \cup \underline{B}_{d-1} \rangle \, \{\underline{B}_d\} \, \{\underline{C}\} \, \{\underline{Q}\}.$$

Let $\mathbf{T}_B$ be denote the following triangulation:

$$\mathbf{T}_B \quad := \quad \langle \underline{A} \rangle \, \langle \underline{B} \rangle \, \{\underline{C}\} \, \{\underline{Q}\} \cup \langle \underline{A} \rangle \, \{\underline{B}\} \, \langle \underline{C} \rangle \, \{\underline{Q}\}$$
$$\text{in cases 1a, 1b,}$$

$$\mathbf{T}_B \quad := \quad \langle \underline{A} \rangle \, \langle \underline{B}_0 \cup \ldots \cup \underline{B}_{d-1} \rangle \, \{\underline{B}_d\} \{\underline{C}\} \, \langle \underline{Q} \rangle \cup \langle \underline{A} \rangle \, \{\underline{B}\} \, \langle \underline{C} \rangle \, \{\underline{Q}\}$$
$$\text{in case 2a,}$$

$$\mathbf{T}_B \quad := \quad \langle \underline{A}_0 \cup \ldots \cup \underline{A}_{l-1} \rangle \, \{\underline{A}_l\} \langle \underline{B} \rangle \, \{\underline{C}\} \, \langle \underline{Q} \rangle \cup \langle \underline{A} \rangle \, \{\underline{B}\} \, \langle \underline{C} \rangle \, \{\underline{Q}\}$$
$$\text{in case 2b,}$$

$$\mathbf{T}_B \quad := \quad \langle \underline{A} \rangle \, \langle \underline{B} \rangle \, \{\underline{C}\} \, \langle \underline{Q} \rangle \cup \langle \underline{A} \rangle \, \{\underline{B}\} \, \langle \underline{C} \rangle \, \{\underline{Q}\}$$
$$\setminus \{\underline{A}_0 \cup \ldots \cup \underline{A}_{l-1}\} \, \{\underline{B}_0 \cup \ldots \cup \underline{B}_{d-1}\} \langle \underline{A}_l \rangle \, \langle \underline{B}_d \rangle \, \{\underline{C}\} \, \{\underline{Q}\}$$
$$\text{in case 2c.}$$

We can write it shorter:

$$\mathbf{T}_B = \langle \underline{A} \rangle \, \langle \underline{B} \rangle \, \{\underline{C}\} \, \{\underline{Q}\} \cup \langle \underline{A} \rangle \, \{\underline{B}\} \, \langle \underline{C} \rangle \, \{\underline{Q}\} \setminus S$$

where

$$S = \begin{cases} \emptyset \quad \text{in cases 1a, 1b, i.e.} Q \in \partial \mathbf{P}^B, \\ \{\underline{A}\} \{\underline{B}_0 \cup \ldots \cup \underline{B}_{d-1}\} \, \langle \underline{B}_d \rangle \, \{\underline{C}\} \, \{\underline{Q}\} \quad \text{in case 2a,} \\ \{\underline{A}_0 \cup \ldots \cup \underline{A}_{l-1}\} \, \langle \underline{A}_l \rangle \, \langle \underline{B} \rangle \, \{\underline{C}\} \, \{\underline{Q}\} \quad \text{in case 2b,} \\ \{\underline{A}_0 \cup \ldots \cup \underline{A}_{l-1}\} \, \langle \underline{A}_l \rangle \, \{\underline{B}_0 \cup \ldots \cup \underline{B}_{d-1}\} \, \langle \underline{B}_d \rangle \, \{\underline{C}\} \, \{\underline{Q}\} \quad \text{in case 2c.} \end{cases}$$

Note that $\mathbf{T}_B = \mathbf{P^E}_{(Q,B_j)}$ for any vertex B_j of $\mathbf{P^E}$. By symmetry let $\mathbf{T}_A = \mathbf{P^E}_{(Q,A_i)}$. Then $\mathbf{T}_A = \langle \underline{A} \rangle \, \langle \underline{B} \rangle \, \{\underline{C}\} \, \{\underline{Q}\} \cup \{\underline{A}\} \, \langle \underline{B} \rangle \, \langle \underline{C} \rangle \, \{\underline{Q}\} \setminus S$

The triangulations $\mathbf{T}_A$ and $\mathbf{T}_B$ differ by an elementary transformation on the basic complex $\mathrm{conv}(\{\underline{A} \, ; \, \underline{B}\} \langle \underline{C} \rangle \{\underline{Q}\})$. Thus it is sufficient to prove the el-equivalence of $\mathbf{T}_B$ and $\mathbf{B}_{(Q)}$, and we get the el-equivalence of $\mathbf{A}_{(Q)}$ and $\mathbf{B}_{(Q)}$ by symmetry.

Note that in cases 1.b and 2.a we have that $\mathbf{T}_B = \mathbf{B}_{(Q)}$. Consider the other cases. Let $\mathbf{T}_s$ denote the following subsets of $\mathrm{SET}^2(V)$.

$$\mathbf{T}_s := \langle \underline{A}_0 \cup \ldots \cup \underline{A}_{l-s} \rangle \, \{\underline{A}_{l-s+1} \cup \ldots \cup \underline{A}_l\} \, \{\underline{B}\} \, \{\underline{C}\}$$

$$\cup \, \langle \underline{A}_0 \cup \ldots \cup \underline{A}_{l-s} \rangle \, \langle \underline{A}_{l-s+1} \cup \ldots \cup \underline{A}_l \rangle \, \{\underline{B}\} \, \{\underline{C}\} \, \{\underline{Q}\}$$

$$\cup \, \{\underline{A}_0 \cup \ldots \cup \underline{A}_{l-s}\} \, \langle \underline{A}_{l-s+1} \cup \ldots \cup \underline{A}_l \rangle \, \langle \underline{B} \rangle \, \{\underline{C}\} \, \{\underline{Q}\}$$

$$\cup \, \{\underline{A}_0 \cup \ldots \cup \underline{A}_{l-s}\} \, \langle \underline{A}_{l-s+1} \cup \ldots \cup \underline{A}_l \rangle \, \{\underline{B}\} \, \langle \underline{C} \rangle \, \{\underline{Q}\} \setminus S.$$

Lemma 4.3. (a) $\mathbf{T}_1 = \mathbf{B}_{(Q)}$.
(b) $\mathbf{T}_s$ *is a triangulation for* $s = 1, \ldots, l$.
(c) $\mathbf{T}_s$ *is el-equivalent to* $\mathbf{T}_{s-1}$ *for* $s = 2, \ldots, l$.

(d) $\mathbf{T}_l$ *is el-equivalent to* $\mathbf{P^E}_{(Q)}$.

Proof. (a) just follows from the definition.

We prove (b) and (c) by induction on s.

Lemma 4.3.1. $\mathbf{P^B}_s := \operatorname{conv}(\underline{A}_0 \cup \ldots \cup \underline{A}_{l-s-1} \cup \underline{B} \cup \underline{C}; \underline{A}_{l-s+1} \cup \ldots \cup \underline{A}_l \cup \underline{Q})$, *where* $s = 1, \ldots, l$, *is a basic polytope.*

Proof. By Lemma 4.2.1, $\operatorname{conv}(\underline{A}_0 \cup \ldots \cup \underline{A}_{l-s-1} \cup \underline{B}; \underline{A}_{l-s+1} \cup \ldots \cup \underline{A}_l \cup \underline{Q}')$ is a basic polytope. Thus

$$(0) \qquad \sum_{j \leq i_{l-s-1}} a_j A_j + \sum b_i B_i = \sum_{j > i_{l-s}} a_j A_j + q Q'$$

where all coefficients are positive. By the definition of Q' and of the projection p

$$(1) \qquad Q' = q'' Q - \sum k_i C_i, \quad \text{where } k_i > 0 \text{ and } q'' > 0.$$

Putting (1) into (0), we get:

$$(2) \qquad \sum_{j \leq i_{l-s-1}} a_j A_j + \sum b_i B_i + \sum q k_i C_i = \sum_{j > i_{l-s}} a_j A_j + q q'' Q.$$

To complete the proof note that (2) is the only relation on the set of vertices. (If we exclude Q we get an affinely independent set.) $\qquad \square$

Lemma 4.3.2.

$$\underline{P}_s^E = \operatorname{conv}(\{\underline{A}_0 \cup \ldots \cup \underline{A}_{l-s-1} \cup \underline{B} \cup \underline{C} \; ; \; \underline{A}_{l-s+1} \cup \ldots \cup \underline{A}_l \cup \underline{Q} \} \langle \underline{A}_{l-s} \rangle \{\underline{C}\})$$

is a set of maximal elementary polytopes such that:

   i) *All the elementary polytopes have the same basic polytope* $\mathbf{P^B}_s$.

   ii) *For any polytope*

$$\mathbf{P}' = \operatorname{conv}(\underline{A}_0 \cup \ldots \cup \underline{A}_{l-s-1} \cup \underline{B} \cup \underline{C} \; ; \; \underline{A}_{l-s+1} \cup \ldots \cup \underline{A}_l \cup \underline{Q} \; : \; *)$$

   in $\underline{P}_s^E$, *where* $* \in \langle \underline{A}_{l-s} \rangle \{\underline{C}\}$, *the canonical triangulation of* $\mathbf{P}'$,

$$\{\underline{A}_0 \cup \ldots \cup \underline{A}_{l-s-1} \cup \underline{B} \cup \underline{C}\} \; \langle \underline{A}_{l-s+1} \cup \ldots \cup \underline{A}_l \cup \underline{Q} \rangle \; \{*\},$$

   is a subcomplex of $\mathbf{T}_s$.

   iii) *For every polytope* a *belonging to* $\operatorname{Star}(\underline{A}_0 \cup \ldots \cup \underline{A}_{l-s-1} \cup \underline{B} \cup \underline{C} : \mathbf{T}_s)$ *there exists a polytope* $\mathbf{E}' \in \underline{P}_s^E$ *such that* $\alpha \subset \mathbf{E}'$.

Proof. As for Lemma 4.1.2. $\qquad \square$

By Lemma 4.0 and Lemma 4.3.2 $\underline{P}_s^E$ determines the elementary transformation which changes

$$\{\underline{A}_0 \cup \ldots \cup \underline{A}_{l-s-1} \cup \underline{B} \cup \underline{C}\} \; \langle \underline{A}_{l-s+1} \cup \ldots \cup \underline{A}_l \cup \underline{Q} \rangle \; \langle \underline{A}_{l-s} \rangle$$

into

$$\langle \underline{A}_0 \cup \ldots \cup \underline{A}_{l-s-1} \cup \underline{B} \cup \underline{C} \rangle \; \{\underline{A}_{l-s+1} \cup \ldots \cup \underline{A}_l \cup \underline{Q}\} \; \langle \underline{A}_{l-s} \rangle$$

transforming the triangulation $\mathbf{T}_s$ to the triangulation $\mathbf{T}_{s+1}$. This shows points a) b) c) of 4.3.

The elementary transformation on the basic complex

$$\operatorname{conv}(\underline{B} \cup \underline{C} \; : \; \underline{Q} \cup \underline{A}_1 \cup \ldots \cup \underline{A}_l) \langle \underline{A}_0 \rangle$$

changes

$$\langle \underline{A}_0 \rangle \{\underline{A}_1 \cup \ldots \cup \underline{A}_l\} \{\underline{B}\} \{\underline{C}\} \; \cup \; \langle \underline{A}_0 \rangle \langle \underline{A}_1 \cup \ldots \cup \underline{A}_l \rangle \{\underline{B}\} \{\underline{C}\} \; \{\underline{Q}\} \subseteq T_l$$

into:

$$\langle \underline{A_0} \rangle \{ \underline{A_1} \cup \ldots \cup \underline{A_l} \} \langle \underline{B} \rangle \{ \underline{C} \} \{ \underline{Q} \}$$
$$\cup \langle \underline{A_0} \rangle \{ \underline{A_1} \cup \ldots \cup \underline{A_l} \} \{ \underline{B} \} \langle \underline{C} \rangle \{ \underline{Q} \} \subseteq \mathbf{P^E}_{(Q)}$$

and transforms $\mathbf{T}_l$ to $\mathbf{P^E}_{(Q)}$.

Thus we have proved Lemma 4.3 and, by symmetry, the whole Proposition 4C. □

5. El-Equivalence of Blow-Ups of a Simplicial Complex

Proposition 5.0. *Let* $\mathbf{K}$ *be simplicial complex, and let* $P, Q \in |\mathbf{K}|$. *Then:*

(a) *The triangulations* $\mathbf{K}_{(Q,P)}$ *and* $\mathbf{K}_{(P,Q)}$ *are el-equivalent*

(b) *The elementary transformations realizing the above equivalence do not change the triangulations outside the simplices which contain both* P *and* Q.

(c) *The elementary transformations are induced (in the sense of section 3) by an elementary transformation of the minimal simplex containing both points* P, Q.

Proof. We consider two cases:

i) P, Q are contained in some simplex Δ of $\mathbf{K}$.

Assume that Δ is minimal i.e. P, Q are not contained in any proper face of Δ. By Proposition 4.A $\Delta_{(P,Q)}$ and $\Delta_{(Q,P)}$ are el-equivalent, and any elementary transformation realizing that does not change the triangulations of proper faces of Δ. By Corollary 3.5, Proposition 3.6 and Proposition 3.8 the triangulations $\mathbf{K}(\Delta_{(P,Q)}) = \mathbf{K}(\Delta)_{(P,Q)} = \mathbf{K}_{(P,Q)}$ and $\mathbf{K}(\Delta_{(Q,P)}) = \mathbf{K}_{(Q,P)}$ are el-equivalent, and the relevant transformations do not change the triangulations outside of CompStar(Δ ; $\mathbf{K}$).

ii) P, Q are not contained in any common simplex.

Hence $P \in \operatorname{int} \Delta^P$ and $Q \in \operatorname{int} \Delta^Q$, for some simplices Δ^P, Δ^Q. Therefore the sets of polytopes of Star(Δ^P ; $\mathbf{K}$) and of Star(Δ^Q ; $\mathbf{K}$) are disjoint. Thus $\mathbf{K}_{(P,Q)} = \mathbf{K}_{(Q,P)}$ □

Lemma 5.1. *Assume* $\mathbf{T}_0, \mathbf{T}_1$ *are two el-equivalent triangulations of* $\mathbf{K}$. *Let* $\mathbf{L}$ *be any common subcomplex of* $\mathbf{T}_0$ *and* $\mathbf{T}_1$ *which is not changed during the corresponding transformations. Let* $P \in |\mathbf{K}|$.

Then $\mathbf{T}_{0(P)}$ *and* $\mathbf{T}_{1(P)}$ *are el-equivalent, and the corresponding transformations do not change the complex* $\mathbf{L}_{(P)}$.

Proof. It suffices to consider the case of $\mathbf{T}_0$, $\mathbf{T}_1$ that differ by an elementary transformation.

Let $\mathbf{S}^E$ denote the elementary subdivision , $\mathbf{C^B}$ the basic complex, $\mathbf{P^B} = \operatorname{conv}(A_0 \ldots A_n ; B_0 \ldots B_m)$ the basic polytope, and

$$\mathbf{P^E}_h = \operatorname{conv}(A_0 \ldots A_n ; B_0 \ldots B_m : C_1^h \ldots C_{i(h)}^h),$$

where $h \in H$, be all the elementary polytopes corresponding to the considered elementary transformation.

By assumption $\mathbf{L}$ is a subcomplex of Ast($\mathbf{P^B}$; $\mathbf{K}$). Consider three cases:

1. $P \notin |\mathbf{C^B}|$. Then $\mathbf{T}_{0(P)}$ and $\mathbf{T}_{1(P)}$ differ by an elementary transformation on the same basic complex $\mathbf{C}^B$. The transformation does not change $\mathbf{L}_{(P)}$.

2. $P \in \mathbf{P^B}$. Then $\mathbf{C}$ is simplicial with respect to $\mathbf{P^B}$. Two canonical triangulations $\mathbf{A}$ and $\mathbf{B}$ of $\mathbf{P^B}$ induce triangulations $\mathbf{T}_0, \mathbf{T}_1$. By Proposition 3.6, $\mathbf{T}_{0(P)}$ and

$\mathbf{T}_{1(P)}$ are induced by $\mathbf{A}_{(P)}$ and $\mathbf{B}_{(P)}$. By Proposition 4.B, $\mathbf{A}_{(P)}$ is el-equivalent to $\mathbf{B}_{(P)}$. By Corollary 3.5 and Proposition 3.8 we obtain our assertion: the corresponding transformations only change the triangulations of polytopes which contain $\mathbf{P}^B$.

3. $P \notin \mathbf{P}^\mathbf{B}$ and $P \in \mathbf{P}^{\mathbf{E}}{}_h$ for some h. This is the same case as 2 above, using Proposition 4.C instead of Proposition 4.B. $\qquad\square$

Proposition 5.2. *Let* Δ *be a simplex and* $P_i \in \Delta$ *for* $i = 1, \ldots, n$. *Assume* $P \in \operatorname{int} \Delta$. *Then*

$$\Delta_{(P_0,\ldots,P_{j-1},P,P_j,\ldots P_l)} \;\; and \;\; \Delta_{(P_0,\ldots,P_{k-1},P,P_k,\ldots P_l)}$$

are el-equivalent for all j, k, l, *and the corresponding transformations do not change the relative boundary of* Δ.

Proof. It is sufficient to consider the transposition

$$\Delta_{(P_0\ldots P_{j-1},P,P_j\ldots P_l)} \;\; \longrightarrow \;\; \Delta_{(P_0,\ldots P_j,P,P_{j+1}\ldots P_l)}$$

Apply Proposition 4.A to $\Delta_{(P_0\ldots P_{j-1})}$ and to the points P_j and P. We get that $\Delta_{(P_0\ldots P_{j-1},P,P_j)}$ and $\Delta_{(P_0\ldots P_{j-1},P_j,P)}$ are el-equivalent and the relevant transformations only change simplices which contain P and P_j. Therefore none of the simplices lying on the boundary is changed. We then get our statement by applying Lemma 5.1 to the points $P_{j+1}, \ldots, P_l$. $\qquad\square$

6. Flat Pairs. Equivalence of Blow-Ups of an Arbitrary Complex

Let $\mathbf{C}$ be a complex in $\mathbf{Q}^d$ and H be an affine space. We denote by $\mathbf{C}_{|H}$ the complex which consists of all faces of $\mathbf{C}$ contained in H. We write $\mathbf{C} \cap H$ for the complex of all the polytopes of the form $\mathbf{P} \cap H$, where $\mathbf{P}$ is a face of $\mathbf{C}$.

Clearly $\mathbf{C}_{|H}$ is a subcomplex of $\mathbf{C} \cap H$. Note also that if $\mathbf{C}$ is simplicial then so is $\mathbf{C}_{|H}$, and if $\mathbf{C}_0$ is a subdivision of $\mathbf{C}_1$ then $\mathbf{C}_0 \cap H$ is also a subdivision of $\mathbf{C}_1 \cap H$.

Theorem 6.1 ([4], Theorem 1 in 5.1). *Every polytope* $\mathbf{P}$ *is affinely isomorphic to the polytope* $\Delta \cap H$, *where* $\Delta \subset \mathbf{R}^k$ *is a simplex and* H *an affine space in* $\mathbf{R}^k$. $\qquad\square$

Although this theorem is proved for the field $\mathbf{R}$, it also holds for $\mathbf{Q}$ and the proof is the same.

Proposition 6.2. *Let* $\mathbf{C}$ *be a complex and let* H *be an affine space. Assume* $P \in |\mathbf{C}| \cap H$. *Then* $\mathbf{C}_{(P)} \cap H = (\mathbf{C} \cap H)_{(P)}$.

Proof. Both operations, blow-up and intersection of complexes, are performed on each polytope separately. Thus it suffices to prove the above for each face $\mathbf{P}$ of $\mathbf{C}$ containing P.

Let Γ be the set of all faces of $\mathbf{P}$ which do not contain P. Let Γ_H be the of all polytopes $\mathbf{K} \cap H$, where $\mathbf{K}$ is a face of $\mathbf{P}$ and $P \notin \mathbf{K} \cap H$. Clearly $\Gamma_H = \{\mathbf{K} \cap H : \mathbf{K} \in \Gamma\}$. Thus

$$
\begin{aligned}
\mathbf{P}_{(P)} &= \{\operatorname{conv}(\mathbf{K}; P) : \mathbf{K} \in \Gamma\} \cup \Gamma, \\
\mathbf{P}_{(P)} \cap H &= \{\operatorname{conv}(\mathbf{K}; P) \cap H : \mathbf{K} \in \Gamma\} \cup \{\mathbf{K} \cap H : \mathbf{K} \in \Gamma\}, \\
&= \{\operatorname{conv}(\mathbf{K} \cap H; P) : \mathbf{K} \in \Gamma\} \cup \Gamma_H, \\
&= \{\operatorname{conv}(\mathbf{K}; P) : \mathbf{K} \in \Gamma_H\} \cup \Gamma_H = (\mathbf{P} \cap H)_{(P)}. \quad\square
\end{aligned}
$$

We say that a pair $(\Delta \; ; \; \{P_i : i = 0, \ldots, l\})$ is *flat with respect to an affine space H* (or *H-flat)* if Δ is a simplex and $H \cap \Delta = \mathrm{conv}(P_i : i = 0, \ldots, l)$ (here the P_i are not necessarily vertices)

Proposition 6.3. *Let $\mathbf{S}$ be a subdivision of the H-flat pair*

$$(\Delta \; ; \; \{P_i : i = 0, \ldots, l\}).$$

Let $\mathbf{P} = \mathrm{conv}(P_1, \ldots, P_l) = \Delta \cap H$. Then $\mathbf{S}_{|H}$ is a subdivision of the pair

$$(\mathbf{P} \; ; \; \{P_i : i = 0, \ldots, l\}).$$

Proof. Let $\Delta = \Delta(A_0, \ldots, A_k)$ and let $Q \in \mathbf{P}$. By induction on $\dim \Delta$ we show the following fact: *For each $Q \in \mathbf{P}$ there exists a polytope $\mathbf{L}^Q \in \mathbf{S}$ such that $Q \in \mathrm{int}\, \mathbf{L}^Q$ and $\mathbf{L}^Q \subseteq \mathbf{P}$* .

This is trivial in dimension one. Now suppose that $\mathbf{L}^Q \not\subseteq \mathbf{P}$ for some $Q \in \mathbf{P}$. By assumption $\mathbf{L}^Q$ is a polytope of the pair $(\Delta \; ; \; \{P_i : i = 0, \ldots, l\})$.
There are two possible cases.

1. $\mathbf{L}^Q = \Delta(A_{i(1)}, \ldots, A_{i(n)})$, where $\{i(1), \ldots, i(n)\} \subset \{1, \ldots, k\}$ and $n < k$. Then $L^Q \cap H = L^Q \cap \mathbf{P}$ is a face of $\mathbf{P}$. Hence $L^Q \cap \mathbf{P} = \mathrm{conv}\{P_i : P_i \in \mathbf{L}^Q\}$. This means that the pair $(\mathbf{L}^Q \; ; \; \{P_i : P_i \in \mathbf{L}^Q\})$ is flat. By the inductive assumption, $Q \in \mathrm{int}\, \mathbf{L}'$ for some polytope $\mathbf{L}' \subseteq \mathbf{L}^Q \cap \mathbf{P}$ of the pair $(\mathbf{L}^Q \; ; \; \{P_i : P_i \in \mathbf{L}^Q\})$. This means that $\mathbf{L}'$ is also a polytope of the pair $(\Delta \; ; \; \{P_i : i = 0, \ldots, l\})$. Finally, by definition, $\mathbf{L}' = \mathbf{L}^Q \subseteq \mathbf{P}$.

2. $\mathbf{L}^Q = \mathrm{conv}(A_{i(1)}, \ldots, A_{i(n)}, P_{j(1)}, \ldots, P_{j(r)})$. Then

$$Q = a_{i(1)}A_{i(1)} + \ldots + a_{i(n)}A_{i(n)} + p_{j(1)}P_{j(1)} + \ldots + p_{j(r)}P_{j(r)},$$

where all coefficients are positive.

Since $Q \in H$, then

$$Q' := (a_{i(1)}/\sum a_{i(j)}) \cdot A_{i(1)} + \ldots + (a_{i(n)}/\sum a_{i(j)}) \cdot A_{i(n)} \text{ is also in } H.$$

But $Q' \in \mathrm{int}\, \Delta(A_{i(1)} \ldots A_{i(n)})$, and the simplex $\Delta(A_{i(1)} \ldots A_{i(n)})$ is a face of $\mathbf{L}^Q$. Now apply 1 to Q' and $\Delta(A_{i(1)}, \ldots, A_{i(n)})$, to get $\Delta(A_{i(1)}, \ldots, A_{i(n)}) \subset H$. Finally

$$L^Q = \mathrm{conv}(A_{i(1)}, \ldots, A_{i(n)}, P_{j(1)}, \ldots P_{j(r)}) \subseteq H.$$

$\square$

Corollary 6.4. *If $\mathbf{S}$ is a subdivision of an H-flat pair $(\Delta \; ; \; \{P_i \; ; \; i = 0, \ldots, 1\}$ then $S_{|H} = S \cap H$.*

Proof. $\mathbf{S}_{|H}$ is a subcomplex of $\mathbf{S} \cap H$. On the other hand, their supports are equal to $|\mathbf{S}_{|H}| = |\mathbf{S}| \cap H = |\mathbf{S} \cap H|$. $\square$

We say that a blow-up $\mathbf{C}_{(P_1, \ldots, P_m)}$ of $\mathbf{C}$ is *complete* iff $\mathrm{Vert}(\mathbf{C}) \subset \{P_1, \ldots, P_m\}$. We say that the blow-up $\mathbf{C}_{(P_1, \ldots, P_m)}$ is *simplicial* if it is a triangulation.

Corollary 6.5. *Let $\mathbf{C}_{(P_1, \ldots, P_m)}$ be a complete blow-up of $\mathbf{C}$. Then $\mathbf{C}_{(P_1, \ldots, P_m)}$ is simplicial.*

Proof. It is sufficient to prove the corollary for arbitrary face $\mathbf{P} \in \mathbf{C}$. Then $\mathbf{P}_{(P_1, \ldots, P_l)} = \mathbf{P}_{(P_{i_1}, \ldots, P_{i_l})}$, where $P_{i_1}, \ldots, P_{i_n}$ is the sequence otained from the sequence $(P_1, \ldots, P_m)$ by removing all points not belonging to $\mathbf{P}$. By Theorem 6.1 $\mathbf{P} = \Delta \cap H$, where Δ is a simplex and H is some affine space. $\mathrm{Vert}(\mathbf{P}) \subset \{P_{i_1}, \ldots, P_{i_l}\}$. Thus the pair $(\Delta; \{P_{i_1}, \ldots, P_{i_l}\})$ is H-flat, and hence by Corollary 6.4 $\mathbf{P}_{(P_{i_1}, \ldots, P_{i_l})} = \Delta_{(P_{i_1}, \ldots, P_{i_l})|H}$ is simplicial. $\square$

Corollary 6.6. *Let* $\mathbf{P}$ *be a polytope in* $\mathbf{Q}^d$ *and* H *be an affine space in* $\mathbf{Q}^d$. *Let* $P_i \in \mathbf{P}$ *for* $i = 0, \ldots, n$ *be such that* $\mathbf{P} \cap H = \mathrm{conv}(P_0, \ldots, P_n)$. *Then*

$$\mathbf{P}_{(P_0,\ldots,P_n)|H} = \mathbf{P}_{(P_0,\ldots,P_n)} \cap H = (\mathbf{P} \cap H)_{(P_0,\ldots,P_n)}.$$

Proof. The second equality follows from Proposition 6.2. Consider the first equality. By Theorem 6.1 we can assume that $\mathbf{P} = \Delta \cap H'$ for some simplex Δ and affine space $H' := \mathrm{aff}(\mathbf{P})$. By replacing H by $H \cap H'$ we can assume that $H \subseteq H'$. Then $\mathbf{P} \cap H = \Delta \cap H$, and the pair $(\Delta ; \{P_0, \ldots, P_n\})$ is H-flat. By Proposition 6.2,

$$\mathbf{P}_{(P_0,\ldots,P_n)} \cap H = (\mathbf{P} \cap H)_{(P_0,\ldots,P_n)} = (\Delta \cap H)_{(P_0,\ldots,P_n)} = \Delta_{(P_0,\ldots,P_n)} \cap H.$$

Since $H \subseteq H'$ we infer that $\Delta_{(P_0,\ldots,P_n)|H}$ is a subcomplex of

$$\Delta_{(P_0,\ldots,P_n)} \cap H'_{|H} = \mathbf{P}_{(P_0,\ldots,P_n)|H}.$$

Since the pair $(\Delta; \{P_0, \ldots, P_n\}$ is flat, Corollary 6.4. yields

$$\Delta_{(P_0,\ldots,P_n)|H} = \Delta_{(P_0,\ldots,P_n)} \cap H.$$

Finally,

$$\mathbf{P}_{(P_0,\ldots,P_n)|H} \supseteq \Delta_{(P_0,\ldots,P_n)|H} = \Delta_{(P_0,\ldots,P_n)} \cap H = \mathbf{P}_{(P_0,\ldots,P_n)} \cap H.$$

The other inclusion is obvious. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Proposition 6.7. *Let* $\mathbf{T}_0$, $\mathbf{T}_1$ *be two el-equivalent triangulations of the* H-*flat pair* $(\Delta ; \Omega)$. *Then* $\mathbf{T}_{0|H}$, $\mathbf{T}_{1|H}$ *are el-equivalent triangulations of the polytope* $\mathbf{P} := \Delta \cap H$.

Proof. It is sufficient to prove the proposition for $\mathbf{T}_0$, $\mathbf{T}_1$ differing by an elementary transformation.

Let $\mathbf{S}^{\mathbf{E}}$ be the elementary subdivision of that transformation, and $\mathbf{P}^{\mathbf{B}}$ its basic polytope. By Definition 2.3, $\mathbf{S}^{\mathbf{E}}$ consists of simplices and of elementary polytopes belonging to $\mathrm{Star}(\mathbf{P}^{\mathbf{B}} ; \mathbf{S}^{\mathbf{E}})$. Consider two cases.

1. $\mathbf{P}^{\mathbf{B}} \subseteq H$. Then $\mathbf{S}^{\mathbf{E}}_{|H}$ consists of simplices of $\mathbf{S}^{\mathbf{E}}$ lying in H, and of the elementary polytopes of $\mathrm{Star}(\mathbf{P}^{\mathbf{B}} ; \mathbf{S}^{\mathbf{E}})$ lying in H. That means that $\mathbf{S}^{\mathbf{E}}_{|H}$ consists of simplices and elementary polytopes belonging to $\mathrm{Star}(\mathbf{P}^{\mathbf{B}} ; \mathbf{S}^{\mathbf{E}}_{|H})$, and thus $\mathbf{S}^{\mathbf{E}}_{|H}$ is an elementary subdivision. By Proposition 6.3, $\mathbf{P} =| \mathbf{T}_{0|H} |=| \mathbf{T}_{1|H} |=| \mathbf{S}^E_{|H} |$. By Corollary 6.4, $\mathbf{T}_{i|H} = \mathbf{T}_i \cap H$ for $i = 0, 1$ are subdivisions of $\mathbf{S}^E_{|H} = \mathbf{S}^{\mathbf{E}} \cap H$. Again by Proposition 6.3, $\mathrm{Vert}(\mathbf{T}_{i|H}) = \mathrm{Vert}(\mathbf{S}^E_{|H}) = \Omega$. Finally $\mathbf{T}_{0|H}$, $\mathbf{T}_{1|H}$ are two different proper triangulations of $\mathbf{S}^{\mathbf{E}}_{|H}$, which means that they differ by an elementary transformation.
2. $\mathbf{P}^{\mathbf{E}}$ is not contained in H. Then none of the elementary polytopes of $\mathbf{S}^{\mathbf{E}}$ lies in H. Therefore $\mathbf{S}^{\mathbf{E}}_{|H}$ consists only of simplices. Hence $\mathbf{T}_{0|H}$, $\mathbf{T}_{1|H}$ are equal, as they are proper triangulations of $\mathbf{S}^{\mathbf{E}}_{|H}$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Proposition 6.8. *Let* $\mathbf{C}_{(P_1,\ldots,P_m)}$ *be a complete blow-up of a complex* $\mathbf{C}$. *Let* $\dim \mathbf{C} = n$. *Let* $P_{i(j)}$, $j = 1, \ldots, l$, *be the sequence obtained from* $P_1, \ldots, P_m$ *by omitting all points belonging to the relative interior of the* n-*dimensional polytopes of* $\mathbf{C}$. *Then* $\mathbf{C}_{(P_{i(1)},\ldots,P_{i(l)})}$ *is also a complete blow-up of* $\mathbf{C}$, *and moreover* $\mathbf{C}_{(P_{i(1)},\ldots,P_{i(l)})}$ *is equivalent to* $\mathbf{C}_{(P_1,\ldots,P_m)}$.

Proof. Let $P_{p(j)}$, $j = 1, \ldots, m - l$, denote the interior points of n-dimensional polytopes in the sequence $P_1, \ldots, P_m$ taken in the same order. We show first that the triangulations $\mathbf{C}_{(P_{i(1)}, \ldots, P_{i(l)}, P_{p(1)}, \ldots, P_{p(m-l)})}$ and $\mathbf{C}_{(P_1, \ldots, P_m)}$ are el-equivalent.

It suffices to show that $\mathbf{C}_{(*, P_{i(h)}, P_{p(g)}, **)}$ and $\mathbf{C}_{(*, P_{p(g)}, P_{i(h)}, **)}$ are el-equivalent for each h and g and arbitrary * and **.

If $P_{i(h)}, P_{p(g)}$ do not belong to any common n-dimensional polytope then both triangulations coincide. Assume that $P_{i(h)}, P_{p(g)}$ belong to a common n-dimensional polytope $\mathbf{P}$. Then the given blow-ups are the same outside $\mathbf{P}$.

Consider the polytope $\mathbf{P}$. By Theorem 6.1 we can write $\mathbf{P} = \Delta \cap H$, where Δ is a simplex and H is an affine space. By assumption of completeness of the blow-up $\mathbf{C}_{(P_1, \ldots, P_m)}$ we have that $\text{vert}(\mathbf{P}) \subset \{P_1, \ldots, P_m\}$. Thus the pair $(\Delta \, ; \, \{P_1, \ldots, P_m\} \cap \mathbf{P})$ is flat. Remove from the sequences $(*, P_{i(h)}, P_{p(g)}, **)$ and $(*, P_{p(g)}, P_{i(h)} **)$ all points which are not in $\mathbf{P}$. Then we get two permutations of the set $\{P_1, \ldots, P_m\} \cap \mathbf{P}$: $(*', P_{i(h)}, P_{p(g)}, **')$ and $(*', P_{p(g)}, P_{i(h)} * *')$ for the respective *' and **'. By Proposition 5.2, $\Delta_{(*', P_{i(h)}, P_{p(g)}, **')}$ and $\Delta_{(*', P_{p(g)}, P_{i(h)}, **')}$ are el-equivalent and the corresponding transformations do not change them on the boundary of Δ. Hence by Corollary 6.6 and Proposition 6.7

$$\Delta_{(*', P_{i(h)}, P_{p(g)}, **')}|H = \mathbf{P}_{(*', P_{i(h)}, P_{p(g)}, **')} = \mathbf{P}_{(*, P_{i(h)}, P_{p(g)}, **)}$$

is el-equivalent to

$$\Delta_{(*', P_{p(g)}, P_{i(h)}, **')}|H = \mathbf{P}_{(*', P_{p(g)}, P_{i(h)}, **')} = \mathbf{P}_{(*, P_{p(g)}, P_{i(h)}, **)}.$$

Moreover the corresponding transformations do not change the triangulations on the boundary of $\mathbf{P}$. Since the triangulations $\mathbf{C}_{(*, P_{i(h)}, P_{p(g)}, **)}$ and $\mathbf{C}_{(*, P_{p(g)}, P_{i(h)}, **)}$ are the same outside of $\mathbf{P}$, the el-equivalence on $\mathbf{P}$ extends to the el-equivalence of these two triangulations. Thus $\mathbf{C}_{(P_{i(1)}, \ldots, P_{i(l)}, P_{p(1)}, \ldots, P_{p(m-l)})}$ and $\mathbf{C}_{(P_1, \ldots, P_m)}$ are el-equivalent. But in the first triangulation we can blow down the points $P_{p(i)}$ for $i = 1, \ldots, m - l$ in the opposite order to get the triangulation $\mathbf{C}_{(P_{i(1)}, \ldots, P_{i(l)})}$. Note that all these blow-downs are done in the category of simplicial complexes because we here operate with complete blow-ups of $\mathbf{C}$ (Corollary 6.5). $\square$

Proposition 6.9. *Any two simplicial blow-ups of an arbitrary complex $\mathbf{C}$ are equivalent.*

Proof. We can assume that the considered blow-ups are complete and have the same set of vertices. (If this is not the case we blow-up the triangulations at the missing points, which does not change the equivalence class). This means that the triangulations are of the form

$$\mathbf{C}_{(Q_{\sigma_0(1)}, \ldots, Q_{\sigma_0(m)})} \quad \text{and} \quad \mathbf{C}_{(Q_{\sigma_1(1)}, \ldots, Q_{\sigma_1(m)})},$$

where σ_0, σ_1 are two permutations of the set $\{1, \ldots, m\}$ and $\{Q_i : i = 1, \ldots, m\} \supseteq \text{Vert}(\mathbf{C})$.

We prove our proposition by induction on $\dim \mathbf{C}$. For $n = 0$ the proposition is obvious. Assume it holds for all k-dimensional complexes where $k < n$, and let $\dim \mathbf{C} = n$. Then $\mathbf{C}$ has a finite number of n-dimensional polytopes, say $\mathbf{P}_j$, for $j = 1, \ldots, w$. In each $\mathbf{P}_j$ we choose a point $P_j \in \text{int} \, \mathbf{P}_j$.

For any permutation σ of the set $\{1, \ldots, m\}$, by $(Q_{\sigma(i(s))} \, ; \, s = 1, \ldots, h)$ where $h \leq m$ we mean the sequence obtained from the sequence $Q_{\sigma(i)}$ by omitting interior points of n-dimensional faces of $\mathbf{C}$. Then by Proposition 6.8 the

complete blow-up $\mathbf{C}_{(Q_{\sigma(i(1))},\dots,Q_{\sigma(i(h))})}$ is equivalent to both $\mathbf{C}_{(Q_{\sigma(1)},\dots,Q_{\sigma(m)})}$ and $\mathbf{C}_{(P_1,\dots,P_w,Q_{\sigma(i(1))},\dots,Q_{\sigma(i(h))})}$. Thus we can we reduce the equivalence of

$$\mathbf{C}_{(Q_{\sigma_0(1)},\dots,Q_{\sigma_0(m)})} \quad \text{and} \quad \mathbf{C}_{(Q_{\sigma_1(1)},\dots,Q_{\sigma_1(m)})}$$

to the equivalence of

$$\mathbf{C}_{(P_1,\dots,P_w,Q_{\sigma_0(i(1))},\dots,Q_{i(\sigma_0(h))})} \quad \text{and} \quad \mathbf{C}_{(P_1,\dots,P_w,Q_{\sigma_1(i(1))},\dots,Q_{\sigma_1(i(h))})},$$

hence to the equivalence of the complete blow-ups of the complex $\mathbf{C}_{(P_1,\dots,P_w)}$ with centers lying in the $(n-1)$-dimensional skeleton $\mathbf{C}(n-1)$ of $\mathbf{C}$.

Note that the complex $\mathbf{C}_{(P_1,\dots,P_w)}$ is simplicial with respect to its subcomplex $\mathbf{C}(n-1)$. By the inductive assumption the blow-ups

$$\mathbf{C}(n-1)_{(Q_{\sigma_0(i(1))}\dots Q_{\sigma_0(i(h))})} \quad \text{and} \quad \mathbf{C}(n-1)_{(Q_{\sigma_1(i(1))}\dots Q_{\sigma_1(1(h))})}$$

are equivalent. By Propositon 3.6 we have:

$$\begin{aligned}
\mathbf{C}_{(P_1,\dots,P_w)}(\mathbf{C}(n-1)_{(Q_{\sigma_0(i(1))}\dots Q_{\sigma_0(i(h))})}) &= \mathbf{C}_{(P_1,\dots,P_w,Q_{\sigma_0(i(1))}\dots Q_{\sigma_0(i(h))})}, \\
\mathbf{C}_{(P_1,\dots,P_w)}(\mathbf{C}(n-1)_{(Q_{\sigma_1(i(1))}\dots Q_{\sigma_1(1(h))})}) &= \mathbf{C}_{(P_1,\dots,P_w,Q_{\sigma_1(i(1))}\dots Q_{\sigma_1(i(h))})}.
\end{aligned}$$

By Corollary 3.5 and the above, the two triangulations are equivalent. □

7. Flat Cuts. Equivalence of All Triangulations of a Complex

Proposition 6.9 allows us to extend the equivalence of triangulations to equivalence of subdivisions: We say that two subdivisions $\mathbf{S}_0$, $\mathbf{S}_1$ of a given complex $\mathbf{C}$ are *equivalent* if some of their simplicial blow-ups are equivalent.

Definition 7.1. Let $\mathbf{C}$ be a complex in $\mathbf{Q}^d$. Let $\mathbf{S}$ be any subdivision of $\mathbf{C}$, and H be a hyperplane in $\mathbf{Q}^d$. The *flat cut* $\mathbf{S} \cdot H$ of $\mathbf{S}$ with H is a subdivision $\mathbf{S}'$ such that every polytope of it is the intersection of some polytope of $\mathbf{S}$ with H, or with one of the halfspaces determined by H.

Proposition 7.2. *Let $\mathbf{S}$ be any subdivision of a complex $\mathbf{C}$. Then $\mathbf{S} \cdot H$ is equivalent to $\mathbf{S}$.*

Proof. Let $\{P_1,\dots,P_k\} = \mathrm{Vert}(\mathbf{S} \cap H)$. We have to show that there exist simplicial blow-ups of $\mathbf{S}$ and of $\mathbf{S} \cdot H$ which are equivalent. We will in fact show that $\mathbf{S}_{(P_1,\dots,P_k)} = (\mathbf{S} \cdot H)_{(P_1,\dots,P_k)}$ is their common blow-up, which can be further blown up to a common simplicial blow-up. By arguments similar to those in Proposition 6.3 we see that $(\mathbf{S} \cdot H)_{(P_1,\dots,P_k)} = (\mathbf{S}_{(P_1,\dots,P_k)}) \cdot H$ It is sufficient to show that $(\mathbf{S}_{(P_1,\dots,P_k)}) \cdot H = \mathbf{S}_{(P_1,\dots,P_k)}$. We prove this for an arbitrary face $\mathbf{P}$ of $\mathbf{S}$. Let $\{P_{i(j)} : j = 1,\dots,n\}$ be those P_i which lie in $\mathbf{P}$. Then $\mathbf{P}_{(P_1,\dots,P_k)} = \mathbf{P}_{(P_{i(1)},\dots,P_{i(n)})}$. From Corollary 6.6 we get

$$\mathbf{P}_{(P_{i(1)},\dots,P_{i(n)})} \cap H = \mathbf{P}_{(P_{i(1)},\dots,P_{i(n)})|H} = \mathbf{P}_{(P_1,\dots,P_k)|H}.$$

That means that intersection with H of every polytope $\mathbf{P}'$ of $\mathbf{P}_{(P_1,\dots,P_k)}$ is a face of $\mathbf{P}'$. Hence this polytope is contained either in H or in one of the halfspaces determined by H. Thus $\mathbf{P}_{(P_1,\dots,P_k)} = \mathbf{P}_{(P_1,\dots,P_k)} \cdot H$ because we get no new polytopes. □

Proposition 7.3. *Let $\mathbf{C}$ be a complex in $\mathbf{Q}^m$. Then all triangulations of $\mathbf{C}$ are equivalent.*

Proof. Let $\mathbf{T}_1$ and $\mathbf{T}_2$ be any two triangulations. Every polytope of $\mathbf{T}_1$ and $\mathbf{T}_2$ is the intersection of a face of $\mathbf{C}$ with some halfspaces in $\mathbf{Q}^m$. Take hyperplanes H_i for $i = 1, \dots, n$ determining all the halfspaces associated with all the faces of $\mathbf{T}_0$ and $\mathbf{T}_1$. We get that

$$\mathbf{T}_0 \cdot H \cdot \dots \cdot H_n = \mathbf{T}_1 \cdot \; H_1 \cdot \dots \cdot H_n.$$

By the previous lemma we are done. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

8. Simplicial Case of Theorem A

Consider a simplicial fan Σ in $\mathbf{Q}^n$. Let $\Sigma(1) = \{\rho_i : i \in I\}$ denote the set of edges of the fan Σ. Attach to each edge $\rho_i \in \Sigma(1)$ any nonzero vector $v_i \in \rho_i$. It will determine the correspondence between cones $\langle v_{i_1}, \dots, v_{i_j} \rangle \in \Sigma$ and simplices $\mathrm{conv}(\rho_{i_1}, \dots, \rho_{i_j})$, which gives rise to a correspondence of fan Σ and the complex

$$\mathbf{C}(\Sigma) := \big\{ \mathrm{conv}(\rho_{i_1}, \dots, \rho_{i_j}) : \langle v_{i_1}, \dots, v_{i_j} \rangle \big\}$$

Any decomposition Σ' of Σ determines a subdivision $\Sigma' \cap \mathbf{C}(\Sigma)$ of $\mathbf{C}(\Sigma)$ and vice versa. Thus we have a natural correspondence f_Σ between *decompositions* (resp. simplicial decompositions of Σ) and *subdivisions and triangulations of the complex* $\mathbf{C}(\Sigma)$. The above correspondence is a bijection. The *inverse map* is given as follows: With a *subdivision* $\mathbf{S}$ of $\mathbf{C}(\Sigma)$ we associate a fan $\{\sigma_i = \mathrm{conv}(t \cdot \Delta_i)$: where $\Delta_i \in \mathbf{S}; t \geq 0)$. The blow-ups and blow-downs of fans are in 1-1 correspondence with the blow-ups and blow-downs of the respective complexes.

Propositions 7.3 can be translated into the following theorem:

Theorem 8.1. *Let $\mathbf{C}$ be an arbitrary complex in $\mathbf{Q}^m$. Let $\mathbf{T}'$, $\mathbf{T}''$ be triangulations of $\mathbf{C}$. Then there exists a sequence of triangulations $\mathbf{T}_i$ of $\mathbf{C}$ for $i = 0, \dots, n$ such that $\mathbf{T}_0 = \mathbf{T}'$, $\mathbf{T}_n = \mathbf{T}''$ and each $\mathbf{T}_{i+1}$ is obtained from $\mathbf{T}_i$ by a blow-up or blow-down.* $\qquad\square$

Note that the above theorem is valid also over the field of real numbers.

Let Σ be a fan in $\mathbf{Q}^n$. By means of the correspondence f_Σ we can translate the above theorem into a theorem on fans:

Theorem 8.2. *Let Σ be a simplicial fan in $\mathbf{Q}^m$. Let Σ', Σ'' be simplicial decompositions of Σ. Then there exists a sequence of simplicial decompositions Σ_i of Σ for $i = 0, \dots, n$ such that, $\Sigma_0 = \Sigma'$, $\Sigma_n = \Sigma''$, and each Σ_{i+1} is obtained from Σ_i by a blow-up or blow-down.* $\qquad\square$

Corollary 8.3. *Let Σ', Σ'' be two simplicial fans such that $|\Sigma'| = |\Sigma''|$. Then there exists a sequence of simplicial fans Σ_i, for $i = 0, \dots, n$, such that $\Sigma_0 = \Sigma'$, $\Sigma_n = \Sigma''$, and each Σ_{i+1} is obtained from Σ_i by a blow-up or blow-down.*

Proof. For any two fans Σ', Σ'' we can easily find a common simplicial subdivision Σ. Then we apply Theorem 8.2. to Σ' and Σ, and to Σ and Σ''. $\qquad\square$

PART 3

Let Σ be a simplicial fan in $\mathbf{Q}^{n+1}$. In the previous section we attached to it the complex $\mathbf{C}(\Sigma) \subset \mathbf{Q}^{n+1}$. Now we shall introduce on $\mathbf{C}(\Sigma)$ an integral structure, which will correspond to the lattice structure. Namely on the set $|\mathbf{C}(\Sigma)|$ we have a function $[\cdot] : \mathbf{C}(\Sigma) \to \mathbf{Q}^{n+1}$ which associates with $P \in |\mathbf{C}(\Sigma)|$ the only primitive vector $[P]$ lying on the ray $t \cdot P$, where $t \geq 0$, and $t \in \mathbf{Q}$. By definition this function

is an injection. We shall call the complex $\mathbf{C}(\Sigma)$ equipped in such a function a *Farey complex* (or F-complex). (See [10].) Using this function allows us to talk of regular triangulations, regular blow-ups, determinants, etc. of simplicial complexes. Namely, a triangulation $\mathbf{T}$ of $\mathbf{C}(\Sigma)$ is regular if coresponding decomposition of Σ is regular. A blow-up of a triangulation is regular if the corresponding blow-up is regular, etc.

9. BASIC DEFINITIONS

Let $V = \mathbf{Q}^k \supset \mathbf{Z}^k$. Let V' be an l-dimensional vector subspace of V with a fixed orientation. Then V' contains the l-dimensional lattice $L = \mathbf{Z}^k \cap V'$.

For vectors $v_1, \ldots, v_k$, in L, by $\det(v_1, \ldots, v_k)$ we mean the determinant of the matrix $(v_1, \ldots, v_k)$, where all v_i are taken in a standard basis of L.

Let $\mathbf{C}$ be a F-complex in $\mathbf{Q}^k$. Let $B_1, \ldots, B_l$ belong to a face of $\mathbf{C}$. Fix an oriented $(l-1)$-dimensional affine subspace H of $\mathbf{Q}^k$, containing $B_1, \ldots, B_l$. Then $V' = \mathrm{aff}(H, 0)$ is an oriented vector subspace of $\mathbf{Q}^k$. Define

$$\det(B_1, \ldots, B_l) := \det([B_1], \ldots, [B_l]).$$

Let $\Delta = \Delta(B_1, \ldots, B_l)$ be a simplex contained in a face of $\mathbf{C}$. Let

$$H = \mathrm{aff}(B_1, \ldots, B_l).$$

Choose an orientation of H. Define

$$\mathrm{Det}\,\Delta := |\det(B_1, \ldots, B_l)|$$

Lemma 9.1 ([1, Proposition 8.2]). *If a simplex $\Delta = \Delta(B_0, \ldots, B_l)$ is not regular there is a point $A \in \Delta$ such that $\mathrm{Det}\,\Delta(A, B_0, \ldots, \check{B}_i, \ldots, B_l) < \mathrm{Det}\,\Delta$ for all i.* □

Lemma 9.2. *Let $\Delta = \Delta(B_0, \ldots, B_l, C_1, \ldots, C_m)$. Set $H = \mathrm{aff}(B_0, \ldots, B_l)$. Then $\mathrm{Det}\,\Delta = \mathrm{Det}\,\Delta(B_0, \ldots, B_l) \cdot f(C_1, \ldots, C_m; H)$, where $f(C_1, \ldots, C_m; H)$ is an integer which depends only on $C_1, \ldots, C_m, H$*

Proof. Let $v_0, \ldots, v_l$ be a basis of the lattice $(\mathrm{aff}\,H, 0)) \cap \mathbf{Z}^{k+1}$. Then

$$\det(B_0, \ldots, B_l, C_1, \ldots, C_m) = \det(B_0, \ldots, B_l) \cdot \det(v_0, \ldots v_l, [C_1], \ldots, [C_m])$$

Set $f(C_1, \ldots, C_m; H) = |\det(v_0, \ldots, v_l, [C_1], \ldots, [C_m])|$. □

Let $\mathbf{K}$ be an F-complex and Δ be a face. The *determinant* $\mathrm{Det}(\mathrm{Star}(\Delta; \mathbf{K}))$ of $\mathrm{Star}(\Delta; \mathbf{K})$ is the maximum of determinants of the simplices in the star.

By the *multiplicative factor* $\mu(\mathrm{Star}(\Delta; \mathbf{K}))$ of $\mathrm{Star}(\Delta; \mathbf{K})$, we mean

$$\mu(\mathrm{Star}(\Delta; \mathbf{K})) = \mathrm{Det}(\mathrm{Star}(\Delta))/\mathrm{Det}\,\Delta,$$

which by 9.2 is a positive integer.

Proposition-Definition 9.3. *Let $\mathbf{K}$ be a complex, simplicial w.r.t a face $\mathbf{P}$. Let $\mathbf{T}'$ be a triangulation of $\mathbf{P}$ and $\Delta \in \mathbf{T}'_{\max}$. Then $\mu(\mathrm{Star}(\Delta'; \mathbf{K}(\mathbf{T}')))$ is independent of Δ' and $\mathbf{T}'$.*

We shall call this number the multiplicative factor of $\mathrm{Star}(\mathbf{P}, \mathbf{K})$ and denote it by $\mu(\mathrm{Star}(\mathbf{P}; \mathbf{K}))$

Proof. Let $\mathbf{P} = \mathrm{conv}(A_0, \ldots, A_l)$. Then

$$\mathrm{Star}(\mathbf{P}, \mathbf{K})_{\max} = \{\mathrm{conv}(A_0, \ldots, A_l, B_1^i, \ldots, B_{s(i)}^i), i = 1, \ldots, r\}.$$

By assumption $(B_1^i, \ldots, B_{s(i)}^i)$ are affinely independent of the points from the affine space $H := \mathrm{aff}\,(A_0, \ldots, A_l)$ (see definition in Section 3).

For $\Delta' = \Delta(A_0', \ldots, A_n')$ we have that

$$\mathrm{Star}(\Delta'; \mathbf{T})_{\max} = \{\Delta(A_0', \ldots, A_n', B_1^i, \ldots, B_{s(i)}^i),\ i = 1, \ldots, r\}.$$

By Lemma 9.2

$$\mathrm{Det}\,\Delta(A_0', \ldots, A_n', B_1^i, \ldots, B_{s(i)}^i) = \mathrm{Det}\,\Delta(A_0', \cdots, A_n') \cdot f(B_1^i, \ldots, B_{s(i)}^i; H).$$

Finally, $\mu(\mathrm{Star}(\Delta', T) = \max\{f(B_1^i, \ldots, B_{s(i)}^i; H)\}.$ □

Fix a $\mathbf{K}$ and $n \in \mathbf{Z}_{\geq 0}$. An (n)-*blow*-up of $\mathbf{K}$ is a blow-up $\mathbf{K}'$ such that all simplices of $\mathbf{K}'$ which are not simplices of $\mathbf{K}$ have determinant $\leq n$. In particular:

Lemma 9.4. *A (1)-blow-up of a regular complex is a regular blow-up.* □

Corollary 9.5. *Let $\mathbf{K}$ be a complex simplicial w.r.t a face $\mathbf{P}$. Let $\mathbf{T}$ be a triangulation of $\mathbf{P}$. Let $Q \in \mathrm{int}\,P$ determines an (n)-blow-up of $\mathbf{T}$. Then Q determines a $(\mu(\mathrm{Star}(\mathbf{P}, \mathbf{K}) \cdot n))$-blow-up of $\mathbf{K}(T)$.* □

Let $\mathbf{T}_0$ and $\mathbf{T}_1$ be triangulations of a complex $\mathbf{K}$ that differ by an elementary transformation at a basic complex $\mathbf{C^B}$. Assume the determinants of all simplices of $\mathbf{T}_{0|\mathbf{C^B}}$ are $\leq r$ and the determinants of all simplices of $\mathbf{T}_{1|\mathbf{C^B}}$ are $\leq s$. Then we say that $\mathbf{T}_0$ and $\mathbf{T}_1$ differ by an elementary transformation of type (r, s) (if $r \geq s$) or (s, r) (if $s \geq r$).

Corollary 9.6. *Let $\mathbf{K}$ be a complex simplicial w.r.t. a face $\mathbf{P}$. Let $\mathbf{T}_0$ and $\mathbf{T}_1$ be two triangulations of $\mathbf{P}$ which differ by elementary transformations of type (r,s). Assume that these transformations do not change triangulations on the boundary of P. Then $\mathbf{K}(\mathbf{T}_0)$ and $\mathbf{K}(\mathbf{T}_1)$ differ by elementary transformations of type $\mu(\mathrm{Star}(\mathbf{P}; \mathbf{K})) \cdot r$ and $\mu(\mathrm{Star}(\mathbf{P}; \mathbf{K})) \cdot s$* □

A *decreasing blow-up* of an F-complex $\mathbf{K}$ is a blow-up such that the determinant of every simplex which is divided is greater than the determinants of the simplices obtained by its subdivision.

By Lemma 9.1, every non-regular simplex or F-complex admits a decreasing blow-up. Note also that decreasing blow-ups do not change any regular simplices.

Lemma 9.7. *Let $\mathbf{K}$ be an F-complex and $P \in |\mathbf{K}|$. Assume that $P \in \Delta$ for $\Delta \in \mathbf{K}$. Then the blow-up of $\mathbf{K}$ at P is decreasing iff it is decreasing for Δ.* □

A *multiple decreasing blow-up* of a given triangulation $\mathbf{T}$ is a triangulation obtained by a composition of finitely many (perhaps zero) decreasing blow-ups of $\mathbf{T}$. By definition a multiple decreasing blow-up does not change any regular simplices.

With a triangulation $\mathbf{T}$ of $\mathbf{K}$ we associate the word

$$\omega(\mathbf{T}) := (\omega_1(\mathbf{T}), \ldots, \omega_i(\mathbf{T}), \ldots)$$

where $\omega_i(\mathbf{T})$ is the number of maximal faces of $\mathbf{K}$ with determinant i. In particular $\omega_i(\mathbf{T}) = 0$ for $i >> 0$. Define

$$\omega^j(\mathbf{T}) = (\omega_j(\mathbf{T}), \omega_{j+1}(\mathbf{T}), \ldots).$$

Let $\mathbf{T}_1$ and $\mathbf{T}_2$ be triangulations of $\mathbf{K}$. Define $\omega(\mathbf{T}_1) > \omega(\mathbf{T}_2)$ if there exists $j \in \mathbf{N}$ such that $\omega_i(\mathbf{T}_1) = \omega_i(\mathbf{T}_2)$ if $i > j$ and $\omega_j(\mathbf{T}_1) > \omega_j(\mathbf{T}_2)$.

Analogously we define $\omega(\mathbf{T}_1) \geq \omega(\mathbf{T}_2)$, $\omega^r(\mathbf{T}_1) > \omega^r(\mathbf{T}_2)$, $\omega^r(\mathbf{T}_1) \geq \omega^r(\mathbf{T}_2)$ for some $r > 0$

Lemma 9.8. *If* $\omega^r(\mathbf{T}_1) \leq \omega^r(\mathbf{T}_2)$ *and* $s \geq r$ *then* $\omega^s(\mathbf{T}_1) \leq \omega^s(\mathbf{T}_2)$. $\qquad\square$

Lemma 9.9. *If* $\mathbf{T}_1$ *is a decreasing blow-up of* $\mathbf{T}_2$ *then* $\omega(\mathbf{T}_1) < \omega(\mathbf{T}_2)$. $\qquad\square$

Lemma 9.10. *If* $\mathbf{T}_1$ *is an* (n)*-blow-up of* $\mathbf{T}_2$ *then* $\omega_{n+1}(\mathbf{T}_1) \leq \omega_{n+1}(\mathbf{T}_2)$. $\qquad\square$

Lemma 9.11. *If* $\mathbf{T}_1$ *and* $\mathbf{T}_2$ *differ by elementary transformation of type (n,n) then* $\omega_{n+1}(\mathbf{T}_1) = \omega_{n+1}(\mathbf{T}_2)$. $\qquad\square$

A *regular decreasing decomposition* of an F-complex $\mathbf{K}$ (or simply *a decreasing decomposition* of $\mathbf{K}$) is a multiple decreasing blow-up which is a regular complex.

Lemma 9.12. *Each F-complex $\mathbf{K}$ admits a decreasing decomposition.*

Proof. By Lemma 9.9 each decreasing blow-up of $\mathbf{T}$ decreases $\omega(\mathbf{T})$. But we cannot decrease this number infinitely many times. $\qquad\square$

We call a triangulation *good* iff any two decreasing decompositions of it can be connected by a finite sequence of regular blow-downs and regular blow-ups. In particular every regular triangulation is good, since it has a unique trivial decreasing decomposition. We say that two good triangulations $\mathbf{T}_0$ and $\mathbf{T}_1$ are *regular equivalent* iff some of their decreasing decompositions can be connected by a finite sequence of regular blow-ups and regular blow-downs. We show in the next sections that all good, and in particular regular, triangulations of a given F-complex $\mathbf{C}$ are regular equivalent.

Lemma 9.13. *A multiple decreasing blow-up of a good triangulation is a good triangulation regular equivalent to the original one.*

Proof. This follows from the fact that each decreasing decomposition of a multiple decreasing blow-up of the triangulation is in fact the decreasing decomposition of that triangulation. $\qquad\square$

10. Relations Between Determinants of Simplices in a Basic Polytope

Lemma 10. *Let* $\mathbf{P}^{\mathbf{B}} := \mathrm{conv}(A_0, \ldots, A_n; B_0, \ldots, B_m)$ *be a basic polytope contained in a face of an F-complex $\mathbf{C}$ in $\mathbf{Q}^k$. Fix an orientation of the affine space* $H = \mathrm{aff}(A_0, \ldots, A_n, B_0, ..., B_m)$. *Then,*

$$\sum (-1)^i \det(A_0, \ldots, \check{A}_i, \ldots, A_n, B_0, \ldots, B_m)[A_i]$$
$$= \sum (-1)^{n+j} \det(A_0, \ldots, A_n, B_0, \ldots, \check{B}_j, \ldots, B_m)[B_j] = g[P],$$

where $P = \Delta(A_0, \ldots, A_n) \cap \Delta(B_0, \ldots, B_m)$ *and g is some integer. Moreover the numbers*

$$\{(-1)^i \, \det(A_0, \ldots, \check{A}_i, \ldots, A_n, B_0, \ldots, B_m);$$
$$(-1)^{n+j} \, \det(A_0, \ldots, A_n, B_0, \ldots, \check{B}_j, \ldots, B_m), g\}$$

are either all positive or all negative.

Proof. Set $V := \mathrm{lin}_{\mathbf{Q}}\{[A_i], [B_j] : i = 0, \ldots, n, j = 0, \ldots, m\}$. The functions

$$\det(\cdot, A_0, \dots, \check{A}_i, \dots, \check{A}_j, \dots, A_n, B_0, \dots, B_m),$$
$$\det(\cdot, A_0, \dots, \check{A}_i, \dots, A_n, B_0, \dots, \check{B}_j, \dots, B_m),$$
$$\det(\cdot, A_0, \dots, A_n, B_0, \dots, \check{B}_i, \dots, \check{B}_j, \dots, B_m)$$

define functionals on V via

$$V \ni w \longrightarrow \det(w, *, \dots, *) := \det(w, [*], \dots, [*]) \in \mathbf{Q}.$$

Set

$$v := \sum (-1)^i \det(A_0, \dots, \check{A}_i, \dots, A_n, B_0, \dots, B_m)[A_i]$$
$$- \sum (-1)^{n+j} \det(A_0, \dots, A_n, B_0, \dots, \check{B}_j, \dots, B_m)[B_j].$$

By definition $v \in V$. The set $\{[A_i], [B_j] : i = 0, \dots, n, j = 0, \dots, m-1\}$ (we omit $j = m$!) is a basis of V (it spans V because $[B_m]$ is a combination of the above vectors, and it is linearly independent because $\dim V = m + n + 1$). Hence we can write for some a_i, b_j

$$v = \sum_{i=0}^{n} a_i[A_i] + \sum_{j=0}^{m-1} b_j[B_j].$$

Then

$$\det(v, A_0, \dots, \check{A}_i, \dots, A_n; B_0, \dots, B_{m-1})$$
$$= a_i \det(A_i, A_0, \dots, \check{A}_i, \dots, A_n, B_0, \dots, B_{m-1})$$
$$= (-1)^i a_i \det(A_0, \dots, A_n, B_0, \dots, B_{m-1}).$$

Thus

$$a_i = (-1)^i \det(v, A_0, \dots, \check{A}_i, \dots, A_n, B_0, \dots, B_{m-1})$$
$$\cdot 1/\det(A_0, \dots, A_n, B_0, \dots, B_{m-1}).$$

Analogously

$$b_j = (-1)^{n+j} \det(v, A_0, \dots, A_n, B_0, \dots, \check{B}_j, \dots, B_{m-1})$$
$$\cdot 1/\det(A_0, \dots, A_n, B_0, \dots, B_{m-1}).$$

By the definition of v,

$$\det(v, A_0, \dots, \check{A}_j, \dots, A_n, B_0, \dots, B_{m-1})$$
$$= \sum (-1)^i \det(A_0, \dots, \check{A}_i, \dots, A_n, B_0, \dots, B_m)$$
$$\cdot \det(A_i, A_0, \dots, \check{A}_j, \dots, A_n, B_0, \dots, B_{m-1})$$
$$- \sum (-1)^{n+i} \det(A_0, \dots, A_n, B_0, \dots, \check{B}_i, \dots, B_m)$$
$$\cdot \det(B_i, A_0, \dots, \check{A}_j, \dots, A_n, B_0, \dots, B_{m-1})$$
$$= (-1)^j \det(A_0, \dots, \check{A}_j, \dots, A_n, B_0, \dots, B_m)(-1)^j$$
$$\cdot \det(A_0, ..., A_n, B_0, ..., B_{m-1})$$
$$- (-1)^{n+m} \det(A_0, \dots, A_n, B_0, \dots, B_{m-1})(-1)^{n+m}$$
$$\cdot \det(A_0, \dots, \check{A}_j, \dots, A_n, B_0, \dots, B_m) = 0.$$

We have used the facts that $\det(A_i, A_0, \ldots \check{A}_j, \ldots, A_n, B_0, \ldots, B_{m-1}) \neq 0$ only for $i = j$ and $\det(B_i, A_0, \ldots A_n, B_0, \ldots, B_{m-1}) \neq 0$ only for $i = m$. Also antisymmetry and bilinearity of determinant have been applied. Analogously,

$$\det(v, A_0, \ldots, A_n, B_0, \ldots, \check{B}_j, \ldots, B_{m-1}) = 0.$$

Consequently by the above $v = 0$.

To show the second part of the assertion, note that

$$(1) \qquad P = \sum_{i=0}^{n} t_i A_i = \sum_{j=0}^{n} s_j B_j \quad \text{for some} \quad t_i, s_j > 0, \quad \sum t_i = \sum s_j = 1.$$

We can treat A_i, B_j as vectors in the standard basis of our lattice $\mathbf{Z}^m$. Then $[A_i] = k_i A_i$ $[B_j] = p_j B_j$, where $k_i, p_j > 0$. Rewrite (1) as,

$$(2) \qquad p[P] = \sum t_i'[A_i] = \sum s_j'[B_j],$$

where $t_i' = t_i/k_i$, $s_j' = s_j/p_j$, and p is the inverse of the sum of the coordinates of the vector $[P]$.

Note that (2) is the only relation (up to a constant factor) between the vectors $[A_i]$, $[B_j]$. Hence the system of numbers

$$((-1)^{n+1} \det(A_0, \ldots, A_n, B_0, \ldots, \check{B}_j, \ldots, B_{m-1}),$$

$$(-1)^i \det(A_0, \ldots, A_i^v, \ldots, A_n, B_0, \ldots, B_m), g)$$

is proportional to the system of numbers (t_i', s_j', p) from (2). Thus we get the assertion. $\qquad\square$

11. Determinants of Simplices of the Pair $(\Delta; \{P, Q\})$

Lemma 11.1. *Let P, Q belong to a simplex $\Delta = \Delta(A_0, \ldots, A_n)$. Set*

$$\mathrm{Det}\,\Delta(A_0, \ldots, A_n) = s,$$

$$\max\{\mathrm{Det}\,\Delta(P, A_0, \ldots, \check{A}_i, \ldots, A_n) : i = 0, \ldots, n\} = r$$

a) *If $P, Q \in \mathrm{int}\,\Delta$ and $\max\{\mathrm{Det}\,\Delta(Q, A_0, \ldots, \check{A}_i, \ldots, A_n) : i = 0, \ldots, n\} \leq s$ then*

$$\max\{\mathrm{Det}\,\Delta(P, Q, A_0, \ldots, \check{A}_i, \ldots, \check{A}_j, \ldots, A_n) : i = 0 \ldots n, j = 0, \ldots, n\} < r.$$

b) *If $\max\{\mathrm{Det}\,\Delta\{Q, A_0, \ldots, \check{A}_i, \ldots, A_n) : i = 0, \ldots, n\} < s$ then*

$$\max\{\mathrm{Det}\,\Delta(P, Q, A_0, \ldots, \check{A}_i, \ldots, \check{A}_j, \ldots, A_n) : i = 0 \ldots n, j = 0, \ldots, n\} < r.$$

Proof. Fix an orientation of the affine hull of the points of Δ. We have representations

$$[P] = \sum \alpha_i^p[A_i], \quad [Q] = \sum \alpha_i^q[A_i] \quad \text{for some} \quad \alpha_i^p, \alpha_i^q \geq 0.$$

Then

$$\det(P, A_0, \ldots, \check{A}_i, \ldots, A_n) = \alpha_i^p \det(A_i, A_0, \ldots, \check{A}_i, \ldots, A_n)$$

$$= (-1)^i \alpha_i^p \det(A_0, \ldots, A_n),$$

$$(*) \qquad \det(Q, A_0, \ldots, \check{A}_i, \ldots, A_n) = (-1)^i \alpha_i^q \det(A_0, \ldots, A_n),$$

$$\mathrm{Det}(Q, A_0, \ldots, \check{A}_i, \ldots, A_n) = \alpha_i^q \mathrm{Det}(A_0, \ldots, A_n),$$

$(**)$

$$\det(P, Q, \ldots, A_0, \ldots, \check{A}_i, \ldots, \check{A}_j, \ldots, A_n)$$
$$= \det(\alpha_i^p[A_i] + \alpha_j^p[A_j], \alpha_i^q[A_j] + \alpha_j^q[A_j], [A_0], \ldots, [\check{A}_i], \ldots, [\check{A}_j], \ldots, [A_n])$$
$$= \alpha_i^p \alpha_j^q \det(A_i, A_j, A_0, \ldots, \check{A}_i, \ldots \check{A}_j, \ldots, A_n)$$
$$+ \alpha_j^p \alpha_i^q \det(A_j, A_i, A_0, \ldots, \check{A}_i, \ldots, \check{A}_j, \ldots, A_n)$$
$$= (\alpha_i^p \alpha_j^q - \alpha_j^p \alpha_i^q) \det(A_i, A_j, A_0, \ldots, \check{A}_i, \ldots, \check{A}_j, \ldots A_n)$$
$$= (\alpha_i^p \alpha_j^q - \alpha_j^p \alpha_i^q)(-1)^{i+j-1} \det(A_0, \ldots, A_n),$$
$$\mathrm{Det}(P, Q, \ldots, A_0, \ldots, \check{A}_i, \ldots, \check{A}_j, \ldots, A_n)$$
$$= |\alpha_i^p \alpha_j^q - \alpha_j^p \alpha_i^q| \, \mathrm{Det}(A_0, \ldots, A_n).$$

(a) By $(*)$) and the assumption on s we claim that $\alpha_i^q \leq 1$ in the representation $[Q] = \sum \alpha_i^q[A]$. On the other hand, $\alpha_i^p, \alpha_j^q > 0$, since $P, Q \in \mathrm{int}\,\Delta$. Thus,

$$\mathrm{Det}\,\Delta(P, Q, A_0, \ldots, \check{A}_i, \ldots, \check{A}_j, \ldots, A_n)$$
$$= |\alpha_i^p \alpha_j^q - \alpha_j^p \alpha_i^q| \, \mathrm{Det}\,\Delta(A_0, \ldots, A_n)$$
$$< \max\{\alpha_i^p \alpha_j^q \, \mathrm{Det}\,\Delta(A_0, \ldots, A_n); \alpha_j^p \alpha_i^q \, \mathrm{Det}\,\Delta(A_0, \ldots, A_n)\}$$
$$\leq \max\{\alpha_j^p \, \mathrm{Det}\,\Delta(A_0, \ldots, A_n); \alpha_i^p \, \mathrm{Det}\,\Delta(A_0, \ldots, A_n)\}$$
$$= \max\{\mathrm{Det}\,\Delta(P, A_0, \ldots, \check{A}_j, \ldots, A_n); \mathrm{Det}\,\Delta(P, A_0, \ldots, \check{A}_i, \ldots, A_n)\} \leq r.$$

(b) By the assumption on s and $(*)$ we have, in particular, $\alpha_i^q < 1$ for all i. Then

$$\mathrm{Det}\,\Delta(P, Q, A_0, \ldots, \check{A}_i, \ldots, \check{A}_j, \ldots, A_n)$$
$$= |\alpha_i^p \alpha_j^q - \alpha_j^p \alpha_i^q| \, \mathrm{Det}\,\Delta(A_0, \ldots, A_n)$$
$$\leq \max\{\alpha_i^p \alpha_j^q \, \mathrm{Det}\,\Delta(A_0, \ldots, A_n); \alpha_j^p \alpha_i^q \, \mathrm{Det}\,\Delta(A_0, \ldots, A_n)\}$$
$$< \max\{\alpha_j^p \, \mathrm{Det}\,\Delta(A_0, \ldots, A_n); \alpha_i^p \, \mathrm{Det}\,\Delta(A_0, \ldots, A_n)\}$$
$$= \max\{\mathrm{Det}\,\Delta(P, A_0, \ldots, \check{A}_j, \ldots, A_n); \mathrm{Det}\,\Delta(P, A_0, \ldots, \check{A}_i, \ldots, A_n)\} \leq r.$$

$\square$

Let $\mathbf{T}$ be a triangulation of an F-complex $\mathbf{K}$. $P, Q \in |T|$.

Lemma 11.2. *If P determines an (n)-blow-up of $\mathbf{T}$ and Q determines a decreasing blow-up of $\mathbf{T}$, then P determines an (n)-blow-up of $\mathbf{T}_{(Q)}$.*

Proof. It suffices to prove the theorem for the respective blow-ups of any simplex $\Delta' \in \mathbf{T}$ such that $P, Q \in \Delta'$. Let $\Delta' = \Delta(A_0 \ldots, A_l)$. By assumption and Lemma 11.1b),

$$\mathrm{Det}\,\Delta(A_0, \ldots, \check{A}_i, \ldots, A_l, P) \leq n$$

implies that

$$\mathrm{Det}\,\Delta(A_0, \ldots, \check{A}_i, \ldots, \check{A}_j, \ldots, A_l, P, Q) \leq n.$$

Finally, $\mathbf{P}$ determines an (n)-blow-up of Δ' and all of $\mathbf{T}$. $\square$

Lemma 11.3. *Assume that P determines an (n)-blow-up of $\mathbf{T}$ which is not decreasing and Q determines a decreasing blow-up of $\mathbf{T}$. Assume $Q \in \mathrm{int}\,\Delta'$ and $P \in \Delta'$ for some $\Delta' \in \mathbf{T}$. Then Q determines an $(n-1)$-blow-up of $\mathbf{T}_{(P)}$.*

Proof. It suffices to prove the theorem for the respective blow-ups of any $\Delta'' \in \mathbf{T}$ such that $Q \in \Delta''$.

Let $\Delta'' = \Delta(A_0, \dots, A_l)$. By the assumptions we have that $P \in \Delta''$. Since P determines an (n)-blow-up of Δ'', but not a decreasing one we claim that $\operatorname{Det} \Delta'' \leq n$. Since Q determines a decreasing blow-up of Δ'', it determines $(n-1)$-blow-up of Δ''. In particular

$$\operatorname{Det} \Delta(A_0, \dots, \check{A}_i, \dots, A_l, Q) \leq n-1, \qquad i = 1, \dots, n.$$

Then by Lemma 11.1b)

$$\operatorname{Det} \Delta(A_0, \dots, \check{A}_i, \dots, \check{A}_j, \dots, A_l, P, Q) \leq n-1.$$

Finally, Q determines an $(n-1)$-blow-up of $\Delta''_{(P)}$ and hence of the whole $\mathbf{T}_{(P)}$. $\square$

Lemma 11.4. *If $\mathbf{P}$ determines an (n)-blow-up of $\mathbf{T}$ which is not decreasing and Q determines a decreasing blow-up of $\mathbf{T}$, then $\mathbf{T}_{(P,Q)}$ and $\mathbf{T}_{(Q,P)}$ differ by a succession of elementary transformations of type $(n, n-1)$.*

Proof. It is sufficient to prove the lemma for the respective blow-ups of any $\Delta' \in \mathbf{T}$ such that $P, Q \in \Delta'$.

Let $\Delta' = \Delta(A_0 \dots, A_l)$. By arguments similar that in the previous proof we claim that $\operatorname{Det} \Delta' \leq n$ and Q determines an $(n-1)$-blow-up of Δ'.

There are three types of simplices of the pair $(\Delta'; P, Q)$:

I. $\Delta'' = \Delta(A_0, \dots, \check{A}_i, \dots, A_l, P)$; then $\operatorname{Det} \Delta'' \leq n$.

II. $\Delta'' = \Delta(A_0, \dots, \check{A}_i, \dots, A_l, Q)$; then $\operatorname{Det} \Delta'' \leq n-1$.

III. $\Delta'' = \Delta(A_0, \dots, \check{A}_i, \dots, \check{A}_j, \dots, A_l, P, Q)$; by Lemma 11.1 b) $\operatorname{Det} \Delta'' \leq n-1$.

By Corollary 4.1.3 we have that each basic polytope in the elementary equivalence of $\Delta'_{(P,Q)}$ and $\Delta'_{(Q,P)}$ is of the form

$$(\mathbf{P^B})' = \operatorname{conv}(A_{i_0}, \dots, A_{i_h}, P; A_{i_{h+1}}, \dots, A_{i_g}, Q)$$

for some subset $\{i_0, \dots, i_g\} \subset \{1, \dots, l\}$. Thus for each elementary transformation one canonical triangulation of the corresponding basic complex $(\mathbf{C^B})'$ consists of simplices of types I and III and the other one consists of simplices of types II and III. Finally, each elementary transformation is of type $(n, n-1)$. $\square$

Lemma 11.5. *If $\mathbf{P}$ determines an (n)-blow-up of $\mathbf{T}$ and Q determines a decreasing (n)-blow-up of $\mathbf{T}$, then $\mathbf{T}_{(P,Q)}$ and $\mathbf{T}_{(P,Q)}$ differ by elementary transformations of type (n, n).*

Proof. Analogously as for Lemma 11.4 $\square$

Lemma 11.6. *Let $n \geq 1$ and $\Delta \in \mathbf{T}$. If $\mathbf{T}$ is regular and $P \in \operatorname{int} \Delta$ determines an (n)-blow-up of $\mathbf{T}$ and $Q \in \operatorname{int} \Delta$ determines a regular blow-up of $\mathbf{T}$, then*

(i) *P determines an $(n-1)$-blow-up of $\mathbf{T}_{(Q)}$,*

(ii) *Q determines an $(n-1)$-blow-up of $\mathbf{T}_{(P)}$,*

(iii) *$\mathbf{T}_{(P,Q)}$ and $\mathbf{T}_{(Q,P)}$ differ by elementary transformations of type $(n-1, n-1)$.*

Proof. Note that if $n = 1$ then $P = Q$ and $\mathbf{T}_{(P)} = \mathbf{T}_{(Q)} = \mathbf{T}_{(P,Q)} = \mathbf{T}_{(Q,P)}$. Assume that $n > 1$. Since Δ and $\mathbf{T}$ are regular, the multiplicative factor of $\operatorname{Star}(\Delta; \mathbf{T})$ equals one. Since all subdivisions and elementary transformations considered in the lemma are induced by subdivisions and elementary transformations of Δ, it suffices

to prove that simplices of any triangulation of the pair $(\Delta; P, Q)$ have determinant $\leq n - 1$.

Let $\Delta = \Delta(A_0, \dots, A_l)$. By assumption we have $[Q] = \sum_{i=0}^{l}[A_i]$. We can assume that P determines an (n)-blow-up of $\mathbf{T}$ but not an $(n-1)$-blow-up. Otherwise we reduce the considerations to smaller n. Without loss of generality P satisfies the relation

$$(1) \qquad [P] = n[A_0] + \cdots + n[A_j] + r_{j+1}[A_{j+1}] + \cdots + r_l[A_l]$$

where $j \geq 0$ and $r_s < n$ for $j + 1 \leq s \leq l$.

By the above

$$\Delta(A_0, \dots \check{A}_i, \dots, A_j, \dots, A_l, P) = n \quad \text{for } i \leq j,$$

$$\text{Det } \Delta(A_0, \dots A_j \dots, \check{A}_i, \dots, A_l, P) = r_i < n \quad \text{for } i > j,$$

$$\text{Det } \Delta(A_0, \dots \check{A}_i, \dots, A_l, Q) = 1 \leq n - 1 \quad \text{for } 0 \leq i \leq l.$$

By Lemma 11.1 a)

$$\text{Det } \Delta(A_0, \dots \check{A}_i, \dots, \check{A}_j, \dots, A_l, P, Q) < n, \qquad 0 \leq i < j \leq l.$$

Let Δ' be a simplex of the pair $(\Delta; P, Q)$ such that Det $(\Delta') = n$. Then by the above $\Delta' = \Delta(A_0, \dots, \check{A}_{i_0}, \dots, A_j, \dots, A_l, P)$ for some $i_0 \leq j$. From (1) we have

$$n[Q] = [P] + (n - r_{j+1})[A_{j+1}] + \cdots + (n - r_l)[A_l],$$

where $(n - r_{j+1}), \dots, (n - r_l)$ are positive. Thus $Q \in \text{conv}(P, A_{j+1}, \dots, A_l) \subset \Delta'$. But Δ' is a simplex of the pair $(\Delta; P, Q)$. Hence Q is a vertex of Δ', which means $Q = P$ and $n = 1$, and we get a contradiction.   □

12. Main Lemma

Lemma 11. *Fix an F-complex $\mathbf{C}$ and an $n \in \mathbf{Z}_{\geq 0}$. Then the following assertions hold.*

A1(n). *If $\mathbf{T}_0$ and $\mathbf{T}_1$ are two good triangulations of $\mathbf{C}$, that differ by an elementary transformation of type (n, n) and if each triangulation $\mathbf{T}$ of $\mathbf{C}$ is good whenever $\omega^{(n+1)}(\mathbf{T}) \leq \omega^{(n+1)}(\mathbf{T}_0)$ or $\omega^{(n+1)}(\mathbf{T}) \leq \omega^{(n+1)}(\mathbf{T}_0)$, then $\mathbf{T}_0$ and $\mathbf{T}_1$ are regular equivalent.*

A2(n). *If $\mathbf{T}_0$ and $\mathbf{T}_1$ are two good triangulations of $\mathbf{C}$, that differ by an elementary transformation of type $(n+1, n)$ and if each triangulation $\mathbf{T}$ of $\mathbf{C}$ is good whenever $\omega^{(n+1)}(\mathbf{T}) \leq \omega^{(n+1)}(\mathbf{T}_0)$ or $\omega^{(n+1)}(\mathbf{T}) \leq \omega^{(n+1)}(\mathbf{T}_0)$, then $\mathbf{T}_0$ and $\mathbf{T}_1$ are regular equivalent.*

B(n). *If a point P determines an (n)-blow-up of a good triangulation $\mathbf{T}$ and if each triangulation $\mathbf{T}'$ of $\mathbf{C}$ is good whenever $\omega^{(n+2)}(\mathbf{T}') \leq \omega^{(n+2)}(\mathbf{T})$, then $\mathbf{T}$ is regular equivalent to $\mathbf{T}_{(P)}$.*

C(n). *If $\mathbf{T}$ is a triangulation all of whose simplices have determinants $\leq n + 1$, then $\mathbf{T}$ is good.*

Proof. We write $\mathbf{T}_0 \sim \mathbf{T}_1$ for two good triangulations which are regular equivalent. We prove the lemma by induction on n.

A(0). For $n = 0$ conditions (a) from A1 and (a) from A2 are satisfied by the empty set. Therefore the assertions are valid automatically.

B(0). Let $P \in \Delta$ for $\Delta \in \mathbf{T}$. If Δ is not regular then Det $\Delta > 1$. Hence the (1)-blow-up at P is decreasing. Thus we can apply Lemma 9.13.

Now suppose that Δ is regular. Since P determines a (1)-blow-up, then by Lemma 9.4 it determines a regular blow-up of Δ. Moreover, by the definition of 1-blow-up and Proposition-Definiton 9.3 all simplices of $\mathrm{Star}(\Delta; \mathbf{T})$ are regular. Consider a decreasing decomposition $\mathbf{T}_{(P_1,\dots,P_k)}$ of $\mathbf{T}$. Since the decomposition does not change any regular simplices, all simplices of $\mathrm{Star}(\Delta; \mathbf{T})$ are preserved. Hence the regular blow-up at P commutes with the decreasing decomposition at the sequence of points $P_1,\dots,P_k$, i.e. $\mathbf{T}_{(P)(P_1,\dots,P_k)} = \mathbf{T}_{(P_1,\dots,P_k,P)}$. The last triangulation is regular equivalent to $\mathbf{T}_{(P_1,\dots,P_k)}$.

Hence we have found an decreasing decomposition of $\mathbf{T}$ which is regular equivalent to the decreasing decomposition of $\mathbf{T}_{(P)}$. This finishes the proof of B(0).

$\mathbf{C}(0)$. Every triangulation $\mathbf{T}$ all of whose simplices have determinants ≤ 1 is a regular triangulation and hence is good.

Proof of the Inductive Step of Lemma 12. We assume that $n > 1$.

Case A. It follows from the assumption and from Lemmas 9.10 and 9.11 that triangulations $\mathbf{T}$ satisfying the condition $\omega_{n+1}(\mathbf{T}) \leq \omega_{n+1}(\mathbf{T}_i)$ where $i = 0, 1$ remain good if $\mathbf{T}$ is transformed by (n)-blow-ups or (n,n)-elementary transformations. Since we consider only these kind of transformations, all the triangulations in the proof below are good.

Let $\mathbf{C^B}$ denote the basic complex of the given elementary transformation, and let $\mathbf{S^E}$ be its elementary subdivision. Without loss of generality we can assume in both cases A1 and A2 that

$$\max\{\mathrm{Det}\,\Delta : \Delta \in \mathbf{T}_{0|\mathbf{C^B}}\} \leq \max\{\mathrm{Det}\,\Delta : \Delta \in \mathbf{T}_{1|\mathbf{C^B}}\}.$$

Let $\mathbf{P^B} = \mathrm{conv}(A_0,\dots,A_n; B_0,\dots,B_m)$ be the basic polytope of the considered elementary transformation. Set $\mathbf{T}_{0|\mathbf{P^B}} = \mathbf{A}$, $\mathbf{T}_{1|\mathbf{P^B}} = \mathbf{B}$. (For the notations $\mathbf{A}, \mathbf{B}$ see section 2). Then

$$M_A := \max\{\mathrm{Det}\,\Delta : \Delta \text{ is a simplex of } \mathbf{A}\},$$

$$M_B := \max\{\mathrm{Det}\,\Delta : \Delta \text{ is a simplex of } \mathbf{B}\}.$$

By Proposition-Definition 9.3 for any triangulation $\mathbf{T}$ of $\mathbf{P^B}$ at any $\Delta \in \mathbf{T}_{\max}$ we have

$$\mathrm{Det}(\mathrm{Star}(\Delta; \mathbf{S^E}(\mathbf{T}))) = \mu \cdot \mathrm{Det}\,\Delta,$$

where $\mu = \mu(\mathrm{Star}(\mathbf{P^B}; \mathbf{S^E}))$ and $\mathbf{S^E}(\mathbf{T})$ is the induced triangulation.

In particular for any $\Delta' \in \mathbf{A}_{\max}$ and $\Delta'' \in \mathbf{B}_{\max}$

$$\mathrm{Det}(\mathrm{Star}(\Delta'; \mathbf{T}_0)) = \mu \cdot \mathrm{Det}\,\Delta',$$

$$\mathrm{Det}(\mathrm{Star}(\Delta''; \mathbf{T}_1)) = \mu \cdot \mathrm{Det}\,\Delta''.$$

It follows from the above that $\mu \cdot M_A \leq \mu \cdot M_B$; hence $M_A \leq M_B$.

Let $A \in \Delta(A_0 \dots A_n)$ satisfy $[A] = \sum[A_i]/q_0$, where q_0 is the GCD of the coordinates.

Let Δ be a maximal simplex of the pair $(\mathbf{P}^B; A)$. Then Δ is of the form

I. $$\Delta(A_0,\dots,\check{A}_i,\dots,A_n,B_0,\dots,B_k),$$

II. $$\Delta(A_0,\dots,\check{A}_i,\dots,\check{A}_j,\dots,A_n,A,B_0,\dots,B_k),$$

III. $$\Delta(A_0,\dots,\check{A}_i,\dots,A_n,A,B_0,\dots,\check{B}_j,\dots,B_k).$$

I. Let $\Delta_{\mathrm{I}} = \Delta(A_0, \dots, \check{A}_i, \dots, A_n, B_0, \dots, B_k)$. Then

$$\det(A, A_0, \dots, \check{A}_i, \dots, A_n, B_0, \dots, \check{B}_j, \dots, B_k)$$
$$= (-1)^i / q_0 \cdot \det(A_0, \dots, A_n, B_0, \dots, \check{B}_j, \dots, B_k).$$

Thus

$$\mathrm{Det}\, \Delta_{\mathrm{I}} = \mathrm{Det}\, \Delta(A, A_0, \dots, \check{A}_i, \dots, A_n, B_0, \dots, \check{B}_j, \dots, B_k) \le M_A.$$

II. Let $\Delta_{\mathrm{II}} = \Delta(A_0, \dots, \check{A}_i, \dots, \check{A}_j, \dots, A_n, A, B_0, \dots, \dots, B_k)$. Then

$$\det(A, A_0, \dots, \check{A}_i, \dots, \check{A}_j, \dots, A_n, B_0, \dots, B_k)$$
$$= 1/q_0 \cdot \det(A_i, A_0, \dots, \check{A}_i, \dots, \check{A}_j, \dots, A_n, B_0, \dots, B_k)$$
$$+ 1/q_0 \cdot \det(A_j, A_0, \dots, \check{A}_i, \dots, \check{A}_j, \dots, A_n, B_0, \dots, B_k)$$
$$= (-1)^i / q_0 \cdot \det(A_0, \dots, \check{A}_j, \dots, A_n, B_0, \dots, B_k)$$
$$- (-1)^j / q_0 \cdot \det(A_0, \dots, \check{A}_i, \dots, A_n, B_0, \dots, B_k)$$
$$= (-1)^{i+j} / q_0 \cdot \{ (-1)^j \cdot \det(A_0, \dots, \check{A}_j, \dots, A_n, B_0, \dots, B_k)$$
$$- (-1)^i \det(A_0, \dots, \check{A}_i, \dots, A_n, B_0, \dots, B_k) \}.$$

By Lemma 10 both last numbers in the braces are negative or both are positive. Thus

$$\mathrm{Det}\, \Delta_{\mathrm{II}} = \mathrm{Det}\, \Delta(A, A_0, \dots, \check{A}_i, \dots, \check{A}_j, \dots, A_n, B_0, \dots, B_k) < M_B.$$

III. Let $\Delta_{\mathrm{III}} = \Delta(A_0, \dots, \check{A}_i, \dots, A_n, B_0, \dots, B_k)$.

Let $P := \Delta(A_0, \dots, A_n) \cap \Delta(B_0, \dots, B_k)$. By Lemma 10, there exists some $p \in \mathbf{Z}$ such that

$$p[P] = \sum (-1)^i \det(A_0, \dots, \check{A}_i, \dots, A_n, B_0, \dots, B_k)[A_i].$$

Moreover, p and $(-1)^i \det(A_0, \dots, \check{A}_i, \dots, A_n, B_0, \dots, B_k)$ have the same sign. Without loss of generality we can assume that they are positive. Hence $p \in \mathbf{N}$ and

$$p[P] = \sum \mathrm{Det}\, \Delta(A_0, \dots, \check{A}_i, \dots, A_n, B_0, \dots, B_k)[A_i].$$

By definition

$$\mathrm{Det}\, \Delta_{\mathrm{III}} = \mathrm{Det}\, \Delta(A_0, \dots, \check{A}_i, \dots, A_n, B_0, \dots, B_k) \le M_B.$$

Assume that $\mathrm{Det}\, \Delta_{\mathrm{III}} = M_B$. By the definition of the point A,

$$q_0 M_B[A] = \sum_j M_B[A_j]$$
$$= p[P] + \sum_j (M_B - \mathrm{Det}\, \Delta(A_0, \dots, \check{A}_j, \dots, A_n, B_0, \dots, B_k))[A_j].$$

Let $k_j := (M_B - \mathrm{Det}\, \Delta(A_0, \dots, \check{A}_j, \dots, A_n, B_0, \dots, B_k))$. Then $k_j \ge 0$. Set $J = \{ j \in \{0, \dots, n\} : k_j > 0 \}$. By the above,

$$(1) \qquad\qquad q_0 M_B[A] = p[P] + \sum_{j \in J} k_j [A_j].$$

By definition $i \notin J$. Thus

$$(2) \qquad\qquad \{ A_j : j \in J \} \subset \Delta(A_0, \dots, \check{A}_i, \dots, A_n, B_0, \dots, B_k).$$

and

$$(3) \qquad P \in \Delta(B_0, \dots, B_k) \subset \Delta(A_0, \dots, \check{A}_i, \dots, A_n, B_0, \dots, B_k)$$

By (1) $A \in \Delta' = \text{conv}(\{P, A_j : j \in J\})$. From (2) and (3) we conclude that $\Delta' \subseteq \Delta_{\text{III}} = \Delta(A_0, \dots, \check{A}_i, \dots, A_n, B_0, \dots, B_k)$. Thus $A \in \Delta_{\text{III}}$, which contradicts the assumption that Δ_{III} is a simplex of $(\mathbf{P^B}; \{A\})$. Finally, $\text{Det}\,\Delta_{\text{III}} < M_B$.

Case A1(n). By the assumption, $M_A \cdot \mu \le M_B \cdot \mu \le n$. By the above considerations the point A determines (M_B)-decomposition of $\mathbf{A}$ and $\mathbf{B}$. Hence by Corollary 9.5 it determines (n)-decomposition of $\mathbf{T}_0 = \mathbf{S}^E(\mathbf{A}_{(A)})$ and $\mathbf{T}_1 = \mathbf{S}^E(\mathbf{B}_{(A)})$. By $\mathbf{B}(n-1)$ we have $\mathbf{T}_0 \sim \mathbf{T}_{0_{(A)}}$ and $\mathbf{T}_1 \sim \mathbf{T}_{1_{(A)}}$.

$\mathbf{A}_{(A)}$ and $\mathbf{B}_{(A)}$ differ by elementary transformations whose basic polytopes are of the form

$$(\mathbf{P^B})' = \text{conv}(A_{i_1}, \dots, A_{i_s}, A; B_0, \dots, B_k, A_{i_{s+1}}, \dots, A_{i_h})$$

where $\{i_1, \dots, i_h\} \subset \{1, \dots, n\}$. Thus one canonical triangulation of the corresponding basic complex of these transformations consists of maximal simplices of types I and II and another one consists of simplices of types II and III. Thus by the above these triangulations differ by elementary transformations of type $(M_B, M_B - 1)$ and it follows from Corollary 9.6 that induced triangulations $\mathbf{S^E}(.)$ differ by elementary transformations of type $(n, n-1)$. We apply A2($n-1$) and get $\mathbf{T}_{0(A)} \sim \mathbf{T}_{1(A)}$. Finally, $\mathbf{T}_0 \sim \mathbf{T}_1$

Case A2(n). By the assumption and since $M_A \le M_B$, we have that $M_B \cdot \mu \le n+1$ and $M_A \cdot \mu \le n$. If $M_A = M_B$ then $M_A \cdot \mu = M_B \cdot \mu \le n$, and we are done by the previous case. Assume $M_A \le M_B - 1$. The point A detemines an $(\{M_B - 1\})$-decomposition of $\mathbf{A}$ and $\mathbf{B}$. Hence it determines (n)-decomposition of $\mathbf{T}_0$ and $\mathbf{T}_1$. By $\mathbf{B}(n-1)$ we have $\mathbf{T}_0 \sim \mathbf{T}_{0_{(A)}}$ and $\mathbf{T}_1 \sim \mathbf{T}_{1_{(A)}}$.

The triangulations $\mathbf{A}_{(A)}$ and $\mathbf{B}_{(A)}$ differ by elementary transformations of type $(M_B - 1, M_B - 1)$, and it follows from Corollary 9.6 that induced triangulations of $\mathbf{S^E}$ differ by elementary transformations of type (n, n). By A1(n) we have $\mathbf{T}_{0(A)} \sim \mathbf{T}_{1(A)}$, and finally $\mathbf{T}_0 \sim \mathbf{T}_1$

Case B(n). All triangulations considered below are good by assumption and by Lemmas 9.10 and 9.11.

We can assume that P determines an $(n+1)$-blow-up which is not decreasing. Otherwise we are done by Lemma 9.13.

Assume $\mathbf{T}$ is not regular. Let $Q \in \text{int}\,\Delta$ for $\Delta \in \mathbf{T}$ determine a decreasing blow-up. If $P \notin \Delta$ then $\Delta \in \mathbf{T}_{(P)}$. Q determines a decreasing blow-up of Δ, and hence by Lemma 9.7 it determines a decreasing blow-up of $\mathbf{T}_{(P)}$. We get $\mathbf{T}_{(P,Q)} \sim \mathbf{T}_{(P)}$ by Lemma 9.13. If $P \in \Delta$, then by Lemma 11.3 Q determines an (n)-blow-up of $\mathbf{T}_{(P)}$, and we get $\mathbf{T}_{(P,Q)} \sim \mathbf{T}_{(P)}$ by B($n-1$). By Lemma 11.4 and A2(n) we get $\mathbf{T}_{(P,Q)} \sim \mathbf{T}_{(Q,P)}$. Finally, $\mathbf{T}_{(Q,P)} \sim \mathbf{T}_{(P)}$, and P determines by Lemma 11.2 an $(n+1)$-blow-up of $\mathbf{T}_{(Q)}$. After finitely many steps we have that $\mathbf{T}'_{(P)} \sim \mathbf{T}_{(P)}$, where $\mathbf{T}'$ is a regular decreasing decomposition of $\mathbf{T}$ and P determines an $(n+1)$-blow-up of $\mathbf{T}'$.

By Lemma 11.6 we find Q such that Q determines a regular blow-up of $\mathbf{T}'$, Q determines an (n)-blow-up of $\mathbf{T}'_{(Q)}$, P determines an (n)-blow-up of $\mathbf{T}_{(Q)}$ and $\mathbf{T}_{(P,Q)}$ and $\mathbf{T}_{(Q,P)}$ differ by elementary transformations of type (n, n). By B($n-1$) we have $\mathbf{T}'_{(P)} \sim \mathbf{T}'_{(P,Q)}$, $\mathbf{T}_{(Q)} \sim \mathbf{T}'_{(Q,P)}$. By A1($n$) we have $\mathbf{T}'_{(P,Q)} \sim \mathbf{T}'_{(Q,P)}$. Finally, $\mathbf{T}'_{(P)} \sim \mathbf{T}'_{(Q)}$. Since Q is the regular blow-up of $\mathbf{T}'$, then $\mathbf{T}'_{(Q)} \sim \mathbf{T}'$.

Since $\mathbf{T}'$ is a regular decomposition of $\mathbf{T}$, then by Lemma 9.13 $\mathbf{T}' \sim \mathbf{T}$. Finally, $\mathbf{T}_{(P)} \sim \mathbf{T}$.

Case C(n). Let Γ^{n+1} denote the set of all triangulations of the complex $\mathbf{C}$ with all simplices having determinants $\leq n+1$. Let Γ_i^{n+1} denote the set of all triangulations of $\mathbf{C}$ with all simplices having determinants $\leq n+1$ and containing at most i maximal simplices with determinants $= n+1$.

We show that all triangulations in Γ^{n+1} are good. By induction on i we prove that all triangulations in Γ_i^{n+1} are good. Then $\bigcup_{i=0}^{\infty} \Gamma_i^{n+1} = \Gamma^{n+1}$ yields the assertion of C(n).

For $i = 0$ we have $\Gamma_0^{n+1} = \Gamma^n$ and we are done by C$(n-1)$. Assume that $i \geq 1$ and all triangulations in Γ_{i-1}^{n+1} are good.

Fix $\mathbf{T} \in \Gamma_i^{n+1}$. Let $\Delta \in \mathbf{T}_{\max}$ and $\mathrm{Det}(\Delta) = n+1$. Let $\mathbf{T}_{(P_1,\ldots,P_r)}$ be a regular decreasing decomposition of $\mathbf{T}$ and s be the smallest index such the $P_s \in \Delta$. Then $\mathbf{T}_{(P_1,\ldots,P_s)} \in \Gamma_{s-1}^{n+1}$ and hence is good. Hence by Lemma 9.13 we have

$$\mathbf{T}_{(P_1,\ldots,P_r)} \sim \mathbf{T}_{(P_1,\ldots,P_s)}.$$

We show that

$$\mathbf{T}_{(P_1,\ldots,P_s)} \sim \mathbf{T}_{(P_s)}.$$

Set $P := P_s$. For $1 \leq j \leq s-1$ we have that $\mathbf{T}_{(P_1,\ldots,P_j,P)}$ is a good triangulation. Since P determines a decreasing blow-up of $\Delta \in \mathbf{T}_{(P_1,\ldots,P_j)}$, by Lemma 9.7 it determines a decreasing blow-up of $\mathbf{T}_{(P_1,\ldots,P_j)}$. Since P_j determines a decreasing blow-up of $\mathbf{T}_{(P_1,\ldots,P_{j-1})}$, it determines the (n)-blow-up. Therefore by Lemma 11.2 P_j determines an (n)-blow-up of $\mathbf{T}_{(P_1,\ldots,P_{j-1},P)}$, and we get by B$(n-1)$ that

$$\mathbf{T}_{(P_1,\ldots,P_{j-1},P)} \sim \mathbf{T}_{(P_1,\ldots,P_{j-1},P,P_j)}.$$

By Lemma 11.5, $\mathbf{T}_{(P_1,\ldots,P_{j-1},P,P_j)}$ differs from $\mathbf{T}_{(P_1\ldots P_j,P)}$ by elementary transformations of type (n,n), which gives by A1(n)

$$\mathbf{T}_{(P_1,\ldots,P_{j-1},P,P_j)} \sim \mathbf{T}_{(P_1,\ldots,P_j,P)}.$$

By the above

$$\mathbf{T}_{(P_1,\ldots,P_j,P)} \sim \mathbf{T}_{(P_1,\ldots,P_{j-1},P)},$$

which gives

$$\mathbf{T}_{(P_1,\ldots,P_{s-1},P)} \sim \mathbf{T}_{(P)}$$

and finally

$$\mathbf{T}_{(P_1,\ldots,P_r)} \sim \mathbf{T}_{(P)}.$$

Let $\mathbf{T}_{(Q_1,\ldots,Q_l)}$ be another regular decomposition. Analogously

$$\mathbf{T}_{(Q_1,\ldots,Q_l)} \sim \mathbf{T}_{(Q)},$$

where $Q \in \Delta$.

Since Q determines an (n)-blow-up of $\mathbf{T}_{(P)}$ and P determines an (n)-blow-up of $\mathbf{T}_{(Q)}$, we get by Lemma 11.2 and B$(n-1)$

$$\mathbf{T}_{(P)} \sim \mathbf{T}_{(P,Q)}, \qquad \mathbf{T}_{(Q)} \sim \mathbf{T}_{(Q,P)}.$$

By Lemma 11.5 $\mathbf{T}_{(P,Q)}$ and $\mathbf{T}_{(Q,P)}$ differ by elementary transformations of type (n,n). Hence by A1(n) we get

$$\mathbf{T}_{(P,Q)} \sim \mathbf{T}_{(Q,P)}.$$

Finally

$$\mathbf{T}_{(P)} \sim \mathbf{T}_{(Q)}.$$

Thus any regular decreasing decompositions of **T** are regular equivalent, which means that **T** is good. This completes the proof of the inductive step in the induction on s. □

13. Main Theorems

Proposition 13.1. *Every triangulation of a Farey complex is good. The blow-up of a triangulation is regular equivalent to the triangulation.*

Proof. Every triangulation satisfies the assumption of $C(n)$ for sufficiently large n. Hence every triangulation is good.

Every blow-up is an (n)-blow-up for sufficiently large n. Therefore it satisfies the conditions in $B(n)$ for a suitable n. Thus by $B(n)$ the triangulations before and after the blow-up are regular equivalent. □

Corollary 13.2. *Any two triangulations of an F-complex* **C** *are regular equivalent. In particular, any two regular triangulations are regular equivalent.*

Proof. From Proposition 13.1 we conclude that triangulations which are equivalent are regular equivalent. By Proposition 7.3 we are done. □

Via the correspondence between a fan Σ and complex $\mathbf{C} = \mathbf{C}(\Sigma)$ described in Sections 8 and 9, Proposition 13.2 can be translated in the following way.

Theorem 13.3. *Let Σ be any simplicial fan in $\mathbf{Q}^m$. Let Σ', Σ'' be two regular decompositions of Σ. Then there exists a sequence of regular decompositions Σ_i of Σ, $i = 0, \ldots, n$, such that $\Sigma_0 = \Sigma'$, $\Sigma_n = \Sigma''$, and each Σ_{i+1} is obtained from Σ_i by a regular blow-up or regular blow-down.* □

Since for each two fans Σ', Σ'' such that $|\Sigma'| = |\Sigma''|$ we can find a common regular decomposition Σ, then by Theorem 13.3. we prove Theorem A.

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